A- LEVEL MATHEMATICS

BRIDGING THE GAP

SUMMER 2024

> “Remember, in Mathematics, revision is more effective when questions are attempted.”

NAME: ___________________________________

YEAR 13

Year 2: A Level Mathematics	
Name:	
Target:	
Current target:	
Self-Assessment: Please identify areas in which you believe are your strong points and those you feel you need to improve on Provide evidence to support your assessment with reference to the content in this booklet.	
Strengths	**Areas for Improvement**

CONTENTS

Pearson Edexcel

A Level Mathematics

Specification

Pearson Edexcel Level 3 Advanced GCE in Mathematics (9MA0)

First teaching from September 2017

First certification from 2018

Issue 4

Summary of Pearson Edexcel Level 3 Advanced GCE in Mathematics Specification Issue 4 changes

Summary of changes made between previous issue and this current issue	**Page number**
Paper 1 and Paper 2: Pure Mathematics, Section 1.1 – Example of proof by exhaustion in guidance replaced with better example.	11
Paper 1 and Paper 2: Pure Mathematics, Section 1.1 – Square root symbol changed to include bar over 2.	11
Paper 1 and Paper 2: Pure Mathematics, Section 2.3 – Text in guidance changed from 'use of a calculator **and** completing the square' to 'use of calculator **or** completing the square' to clarify wording.	12
Paper 1 and Paper 2: Pure Mathematics, Section 2.4 – Text in guidance changed from '**The quadratic** may involve powers of 2...' to '**This** may involve powers of 2...' for clarity.	12
Paper 1 and Paper 2: Pure Mathematics, Section 2.5 – Three inequality symbols in guidance changed to not bold.	13
Paper 1 and Paper 2: Pure Mathematics, Section 2.6 – Text in the guidance changed to clarify that roots can be fractional.	13
Paper 1 and Paper 2: Pure Mathematics, Section 2.7 – 'graphs' changed to 'graph'.	14
Paper 1 and Paper 2: Pure Mathematics, Section 2.7 – Text 'Direct proportion between two variables' removed from guidance to avoid misinterpretation that it suggests **only** direct proportion is included.	14
Paper 1 and Paper 2: Pure Mathematics, Section 3.3 – Fixed typo – deleted '5' before 'describes'.	17
Paper 1 and Paper 2: Pure Mathematics, Section 4.2 – Bracket in equation changed from italic to roman.	18
Paper 1 and Paper 2: Pure Mathematics, Section 5.3 – $\pi/2$ removed from list of exact values required for tan, to match DfE content.	19
Paper 1 and Paper 2: Pure Mathematics, Section 5.5 – Inserted space before 'and'.	20
Paper 1 and Paper 2: Pure Mathematics, Section 6.4 – Plus sign made not subscript.	21
Paper 1 and Paper 2: Pure Mathematics, Section 6.4 – Fixed typo – deleted inequality symbol from '$k \leq \log_a x$'.	21
Paper 1 and Paper 2: Pure Mathematics, Section 6.7 – Deleted 'second' from 'Consideration of a second improved model may be required' to make expectation clearer.	22
Paper 1 and Paper 2: Pure Mathematics, Section 7.1 – First sentence of guidance made bold.	23
Paper 1 and Paper 2: Pure Mathematics, Section 7.3 – Line removed from content and guidance column.	23

Paper 1 and Paper 2: Pure Mathematics, Section 7.6 – Text 'which may include direct proportion' deleted from guidance. Example added.	24
Paper 1 and Paper 2: Pure Mathematics, Section 8.2 – Fixed typo – deleted 'x' from end of first sentence in guidance.	24
Paper 1 and Paper 2: Pure Mathematics, Section 8.3 – Two parts of guidance text made bold.	25
Paper 1 and Paper 2: Pure Mathematics, Section 8.5 – Integration by substitution text deleted from guidance as it was a repetition of text in content column.	25
Paper 1 and Paper 2: Pure Mathematics, Section 10.1 – Full stop made not bold.	27
Paper 1 and Paper 2: Pure Mathematics, Section 10.2 – Vector 'a' changed to roman.	27
Paper 1 and Paper 2: Pure Mathematics, Section 10.4 – Formula for distance between two points in three dimensions added to guidance (not bold).	27
Paper 1 and Paper 2: Pure Mathematics, Section 10.5 – Text about use of ratio theorem deleted to avoid confusion over 'ratio theorem'.	28
Assessment information – 'Mathematical Formulae and Statistical Tables' italicised.	29
Synoptic assessment – 'assesses' corrected to 'assess'.	29
Paper 3: Statistics and Mechanics, Section 2.2 – Paragraphs reordered, some text deleted to avoid repetition.	31
Paper 3: Statistics and Mechanics, Section 5.2 – Line deleted from across Content and Guidance columns.	35
Paper 3: Statistics and Mechanics, Section 6.1 – Guidance made bold.	35
Paper 3: Statistics and Mechanics, Section 7.1 – Guidance made bold.	36
Paper 3: Statistics and Mechanics, Section 7.3 – Sentence starting 'Understand and use *suvat* formulae...' made not bold and moved down a row.	36
Paper 3: Statistics and Mechanics, Section 7.4 – 'r', 'v', 'a' and 't' italicised in integrals.	36
Paper 3: Statistics and Mechanics, Section 7.4 – Cross-reference corrected in guidance.	36
Paper 3: Statistics and Mechanics, Section 8.2 – Order of paragraphs changed in guidance.	37
Paper 3: Statistics and Mechanics, Section 8.3 – Section starting 'The inverse square law...' moved from guidance to content, as it is part of DfE wording.	37
Paper 3: Statistics and Mechanics, Section 8.4 – Text 'restricted to forces in two perpendicular directions or simple cases of forces given as 2-D vectors' added to content to match DfE wording.	38
Breakdown of Assessment Objectives – Ranges for AO2 and AO3 changed.	41

Earlier issues show previous changes.

If you need further information on these changes or what they mean, contact us via our website at: qualifications.pearson.com/en/support/contact-us.html.

Contents

1 Introduction

Why choose Edexcel A Level Mathematics?

We have listened to feedback from all parts of the mathematics subject community, including higher education. We have used this opportunity of curriculum change to redesign a qualification that reflects the demands of a wide variety of end users, as well as retaining many of the features that have contributed to the increasing popularity of GCE Mathematics in recent years.

We will provide:

- **Simple, intuitive specifications** that enable co-teaching and parallel delivery. Increased pressure on teaching time means that it's important you can cover the content of different specifications together. Our specifications are designed to help you co-teach A and AS Level, as well as deliver maths and further maths in parallel.
- **Clear, familiar, accessible exams**. Our new exam papers will deliver everything you'd expect from us as the leading awarding body for maths. They'll take the most straightforward and logical approach to meet the government's requirements. They'll use the same clear design that you've told us makes them so accessible, while also ensuring a range of challenge for all abilities.
- **A wide range of exam practice** to fully prepare students and help you track progress. With the new linear exams your students will want to feel fully prepared and know how they're progressing. We'll provide lots of exam practice to help you and your students understand and prepare for the assessments, including secure mock papers, practice papers and free topic tests with marking guidance.
- **Complete support and free materials** to help you understand and deliver the specification. Change is easier with the right support, so we'll be on hand to listen and give advice on how to understand and implement the changes. Whether it's through our Launch, Getting Ready to Teach, and Collaborative Networks events or via the renowned Maths Emporium, we'll be available face to face, online or over the phone throughout the lifetime of the qualification. We'll also provide you with free materials such as schemes of work, topic tests and progression maps.
- **The published resources you know and trust**, fully updated for 2017. Our new A Level Maths and Further Maths textbooks retain all the features you know and love about the current series, while being fully updated to match the new specifications. Each textbook comes packed with additional online content that supports independent learning and they all tie in with the free qualification support, giving you the most coherent approach to teaching and learning.

Supporting you in planning and implementing this qualification

Planning

- Our **Getting Started** guide gives you an overview of the new A Level qualification to help you to get to grips with the changes to content and assessment, as well as helping you understand what these changes mean for you and your students.
- We will give you a **course planner** and **scheme of work** that you can adapt to suit your department.
- **Our mapping documents** highlight the content changes between the legacy modular specification and the new linear specifications.

Teaching and learning

There will be lots of free teaching and learning support to help you deliver the new qualifications, including:

- topic guides covering new content areas
- teaching support for problem solving, modelling and the large data set
- a student guide containing information about the course to inform your students and their parents.

Preparing for exams

We will also provide a range of resources to help you prepare your students for the assessments, including:

- specimen papers written by our senior examiner team
- practice papers made up from past exam questions that meet the new criteria
- secure mock papers
- marked exemplars of student work with examiner commentaries.

ResultsPlus and exam Wizard

ResultsPlus provides the most detailed analysis available of your students' exam performance. It can help you identify the topics and skills where further learning would benefit your students.

Exam Wizard is a data bank of past exam questions (and sample paper and specimen paper questions) allowing you to create bespoke test papers.

Get help and support

Mathematics Emporium – support whenever you need it

The renowned Mathematics Emporium helps you keep up to date with all areas of maths throughout the year, as well as offering a rich source of past questions and, of course, access to our in-house maths experts Graham Cumming and his team.

Sign up to get Emporium emails

Get updates on the latest news, support resources, training and alerts for entry deadlines and key dates direct to your inbox. Just email mathsemporium@pearson.com to sign up.

Emporium website

Over 12 000 documents relating to past and present Edexcel mathematics qualifications available free. Visit www.edexcelmaths.com to register for an account.

Qualification at a glance

Content and assessment overview

The Pearson Edexcel Level 3 Advanced GCE in Mathematics consists of three externally-examined papers.

Students must complete all assessment in May/June in any single year.

Paper 1: Pure Mathematics 1 (*Paper code: 9MA0/01) **Paper 2: Pure Mathematics 2 (*Paper code: 9MA0/02)**
Each paper is: ***2-hour written examination*** ***33.33% of the qualification*** ***100 marks***
Content overview • Topic 1 – Proof • Topic 2 – Algebra and functions • Topic 3 – Coordinate geometry in the (x, y) plane • Topic 4 – Sequences and series • Topic 5 – Trigonometry • Topic 6 – Exponentials and logarithms • Topic 7 – Differentiation • Topic 8 – Integration • Topic 9 – Numerical methods • Topic 10 – Vectors
Assessment overview • Paper 1 and Paper 2 may contain questions on any topics from the Pure Mathematics content. • Students must answer all questions. • Calculators can be used in the assessment.

Paper 3: Statistics and Mechanics (*Paper code: 9MA0/03)
2-hour written examination ***33.33% of the qualification*** ***100 marks***
Content overview **Section A: Statistics** • Topic 1 – Statistical sampling • Topic 2 – Data presentation and interpretation • Topic 3 – Probability • Topic 4 – Statistical distributions • Topic 5 – Statistical hypothesis testing **Section B: Mechanics** • Topic 6 – Quantities and units in mechanics • Topic 7 – Kinematics • Topic 8 – Forces and Newton's laws • Topic 9 – Moments
Assessment overview • Paper 3 will contain questions on topics from the Statistics content in Section A and Mechanics content in Section B. • Students must answer all questions. • Calculators can be used in the assessment.

*See *Appendix 8: Codes* for a description of this code and all other codes relevant to this qualification.

2 Subject content and assessment information

Qualification aims and objectives

The aims and objectives of this qualification are to enable students to:

- understand mathematics and mathematical processes in a way that promotes confidence, fosters enjoyment and provides a strong foundation for progress to further study
- extend their range of mathematical skills and techniques
- understand coherence and progression in mathematics and how different areas of mathematics are connected
- apply mathematics in other fields of study and be aware of the relevance of mathematics to the world of work and to situations in society in general
- use their mathematical knowledge to make logical and reasoned decisions in solving problems both within pure mathematics and in a variety of contexts, and communicate the mathematical rationale for these decisions clearly
- reason logically and recognise incorrect reasoning
- generalise mathematically
- construct mathematical proofs
- use their mathematical skills and techniques to solve challenging problems that require them to decide on the solution strategy
- recognise when mathematics can be used to analyse and solve a problem in context
- represent situations mathematically and understand the relationship between problems in context and mathematical models that may be applied to solve them
- draw diagrams and sketch graphs to help explore mathematical situations and interpret solutions
- make deductions and inferences and draw conclusions by using mathematical reasoning
- interpret solutions and communicate their interpretation effectively in the context of the problem
- read and comprehend mathematical arguments, including justifications of methods and formulae, and communicate their understanding
- read and comprehend articles concerning applications of mathematics and communicate their understanding
- use technology such as calculators and computers effectively and recognise when their use may be inappropriate
- take increasing responsibility for their own learning and the evaluation of their own mathematical development.

Overarching themes

The overarching themes should be applied along with associated mathematical thinking and understanding, across the whole of the detailed content in this specification.

These overarching themes are inherent throughout the content and students are required to develop skills in working scientifically over the course of this qualification. The skills show teachers which skills need to be included as part of the learning and assessment of the students.

Overarching theme 1: Mathematical argument, language and proof

A Level Mathematics students must use the mathematical notation set out in the booklet *Mathematical Formulae and Statistical Tables* and be able to recall the mathematical formulae and identities set out in *Appendix 1*.

	Knowledge/skill
OT1.1	Construct and present mathematical arguments through appropriate use of diagrams; sketching graphs; logical deduction; precise statements involving correct use of symbols and connecting language, including: constant, coefficient, expression, equation, function, identity, index, term, variable.
OT1.2	Understand and use mathematical language and syntax as set out in the content.
OT1.3	Understand and use language and symbols associated with set theory, as set out in the content. Apply to solutions of inequalities and probability.
OT1.4	Understand and use the definition of a function; domain and range of functions.
OT1.5	Comprehend and critique mathematical arguments, proofs and justifications of methods and formulae, including those relating to applications of mathematics.

Overarching theme 2: Mathematical problem solving

	Knowledge/skill
OT2.1	Recognise the underlying mathematical structure in a situation and simplify and abstract appropriately to enable problems to be solved.
OT2.2	Construct extended arguments to solve problems presented in an unstructured form, including problems in context.
OT2.3	Interpret and communicate solutions in the context of the original problem.
OT2.4	Understand that many mathematical problems cannot be solved analytically, but numerical methods permit solution to a required level of accuracy.
OT2.5	Evaluate, including by making reasoned estimates, the accuracy or limitations of solutions, including those obtained using numerical methods.
OT2.6	Understand the concept of a mathematical problem-solving cycle, including specifying the problem, collecting information, processing and representing information and interpreting results, which may identify the need to repeat the cycle.
OT2.7	Understand, interpret and extract information from diagrams and construct mathematical diagrams to solve problems, including in mechanics.

Overarching theme 3: Mathematical modelling

	Knowledge/skill
OT3.1	Translate a situation in context into a mathematical model, making simplifying assumptions.
OT3.2	Use a mathematical model with suitable inputs to engage with and explore situations (for a given model or a model constructed or selected by the student).
OT3.3	Interpret the outputs of a mathematical model in the context of the original situation (for a given model or a model constructed or selected by the student).
OT3.4	Understand that a mathematical model can be refined by considering its outputs and simplifying assumptions; evaluate whether the model is appropriate.
OT3.5	Understand and use modelling assumptions.

Use of data in statistics

Pearson has provided a large data set, which will support the assessment of Statistics in Paper 3: Statistics and Mechanics. Students are required to become familiar with the data set in advance of the final assessment.

Assessments will be designed in such a way that questions assume knowledge and understanding of the data set. The expectation is that these questions should be likely to give a material advantage to students who have studied and are familiar with the data set. They might include questions/tasks that:

- assume familiarity with the terminology and contexts of the data, and do not explain them in a way that gives students who have not studied the data set the same opportunities to access marks as students who have studied them
- use summary statistics or selected data from, or statistical diagrams based on, the data set – these might be provided in the question or task, or as stimulus materials
- are based on samples related to the contexts in the data set, where students' work with the data set will help them understand the background context and/or
- require students to interpret data in ways that would be too demanding in an unfamiliar context.

Students will not be required to have copies of the data set in the examination, nor will they be required to have detailed knowledge of the actual data within the data set.

The data set can be downloaded from our website, qualifications.pearson.com. This data set should be appropriate for the lifetime of the qualification. However we will review the data set on an annual basis to ensure it is appropriate. If we need to make changes to the data set, we will notify centres before the beginning of the two-year course before students complete their examination.

Paper 1 and Paper 2: Pure Mathematics

To support the co-teaching of this qualification with the AS Mathematics qualification, common content has been highlighted in bold.

Topics	What students need to learn:		
	Content		Guidance
1 **Proof**	1.1	**Understand and use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion; use methods of proof, including:**	**Examples of proofs:**
		Proof by deduction	**Proof by deduction** **e.g. using completion of the square, prove that $n^2 - 6n + 10$ is positive for all values of n or, for example, differentiation from first principles for small positive integer powers of x** or proving results for arithmetic and geometric series. **This is the most commonly used method of proof throughout this specification**
		Proof by exhaustion	**Proof by exhaustion** **Given that p is a prime number such that $3 < p < 25$, prove by exhaustion, that $(p - 1)(p + 1)$ is a multiple of 12.**
		Disproof by counter example	**Disproof by counter example** **e.g. show that the statement "$n^2 - n + 1$ is a prime number for all values of n" is untrue**
		Proof by contradiction (including proof of the irrationality of $\sqrt{2}$ and the infinity of primes, and application to unfamiliar proofs).	

Topics	What students need to learn:		
		Content	**Guidance**
2 **Algebra and functions**	2.1	**Understand and use the laws of indices for all rational exponents.**	$a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, $(a^m)^n = a^{mn}$ **The equivalence of** $a^{\frac{m}{n}}$ **and** $\sqrt[n]{a^m}$ **should be known.**
	2.2	**Use and manipulate surds, including rationalising the denominator.**	**Students should be able to simplify algebraic surds using the results** $\left(\sqrt{x}\right)^2 = x$, $\sqrt{xy} = \sqrt{x}\sqrt{y}$ **and** $\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right) = x - y$
	2.3	**Work with quadratic functions and their graphs.** **The discriminant of a quadratic function, including the conditions for real and repeated roots.** **Completing the square.** **Solution of quadratic equations** **including solving quadratic equations in a function of the unknown.**	**The notation** $\mathrm{f}(x)$ **may be used** **Need to know and to use** $b^2 - 4ac > 0$, $b^2 - 4ac = 0$ **and** $b^2 - 4ac < 0$ $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$ **Solution of quadratic equations by factorisation, use of the formula, use of a calculator or completing the square.** **These functions could include powers of** x**, trigonometric functions of** x**, exponential and logarithmic functions of** x**.**
	2.4	**Solve simultaneous equations in two variables by elimination and by substitution, including one linear and one quadratic equation.**	**This may involve powers of 2 in one unknown or in both unknowns,** **e.g. solve** $y = 2x + 3$, $y = x^2 - 4x + 8$ **or** $2x - 3y = 6$, $x^2 - y^2 + 3x = 50$

Topics	What students need to learn:		
		Content	**Guidance**
2 **Algebra and functions** *continued*	2.5	**Solve linear and quadratic inequalities in a single variable and interpret such inequalities graphically,**	**e.g. solving** $ax + b > cx + d$, $px^2 + qx + r \geq 0$, $px^2 + qx + r < ax + b$ **and interpreting the third inequality as the range of** x **for which the curve** $y = px^2 + qx + r$ **is below the line with equation** $y = ax + b$
		including inequalities with brackets and fractions.	**These would be reducible to linear or quadratic inequalities** **e.g.** $\frac{a}{x} < b$ **becomes** $ax < bx^2$
		Express solutions through correct use of 'and' and 'or', or through set notation.	**So, e.g.** $x < a$ **or** $x > b$ **is equivalent to** $\{ x : x < a \} \cup \{ x : x > b \}$ **and** $\{ x : c < x \} \cap \{ x : x < d \}$ **is equivalent to** $x > c$ **and** $x < d$
		Represent linear and quadratic inequalities such as $y > x + 1$ **and** $y > ax^2 + bx + c$ **graphically.**	**Shading and use of dotted and solid line convention is required.**
	2.6	**Manipulate polynomials algebraically, including expanding brackets and collecting like terms, factorisation and simple algebraic division; use of the factor theorem.**	**Only division by** $(ax + b)$ **or** $(ax - b)$ **will be required.** **Students should know that if** $f(x) = 0$ **when** $x = \frac{b}{a}$**, then** $(ax - b)$ **is a factor of** $f(x)$**.** **Students may be required to factorise cubic expressions such as** $x^3 + 3x^2 - 4$ **and** $6x^3 + 11x^2 - x - 6$**.**
		Simplify rational expressions, including by factorising and cancelling, and algebraic division (by linear expressions only).	Denominators of rational expressions will be linear or quadratic, e.g. $\frac{1}{ax+b}$, $\frac{ax+b}{px^2+qx+r}$, $\frac{x^3+a^3}{x^2-a^2}$

Topics	What students need to learn:		
	Content		Guidance
2 **Algebra and functions** *continued*	2.7	**Understand and use graphs of functions; sketch curves defined by simple equations including polynomials**	**Graph to include simple cubic and quartic functions,** **e.g. sketch the graph with equation $y = x^2(2x - 1)^2$**
		The modulus of a linear function.	Students should be able to sketch the graph of $y = \lvert ax + b \rvert$ They should be able to use their graph. For example, sketch the graph with equation $y = \lvert 2x - 1 \rvert$ and use the graph to solve the equation $\lvert 2x - 1 \rvert = x$ or the inequality $\lvert 2x - 1 \rvert > x$
		$y = \frac{a}{x}$ **and** $y = \frac{a}{x^2}$ **(including their vertical and horizontal asymptotes)**	**The asymptotes will be parallel to the axes e.g. the asymptotes of the curve with equation $y = \frac{2}{x+a} + b$ are the lines with equations $y = b$ and $x = -a$**
		Interpret algebraic solution of equations graphically; use intersection points of graphs to solve equations.	
		Understand and use proportional relationships and their graphs.	**Express relationship between two variables using proportion "$\propto$" symbol or using equation involving constant** **e.g. the circumference of a semicircle is directly proportional to its diameter so $C \propto d$ or $C = kd$ and the graph of C against d is a straight line through the origin with gradient k.**

Topics	What students need to learn:		
	Content		Guidance
2 **Algebra and functions** *continued*	2.8	Understand and use composite functions; inverse functions and their graphs.	The concept of a function as a one-one or many-one mapping from $\mathbb{R}$ (or a subset of $\mathbb{R}$) to $\mathbb{R}$. The notation $\mathrm{f} : x \mapsto$ and $\mathrm{f}(x)$ will be used. Domain and range of functions. Students should know that fg will mean 'do g first, then f' and that if f^{-1} exists, then $\mathrm{f}^{-1}\mathrm{f}(x) = \mathrm{ff}^{-1}(x) = x$ They should also know that the graph of $y = \mathrm{f}^{-1}(x)$ is the image of the graph of $y = \mathrm{f}(x)$ after reflection in the line $y = x$
	2.9	**Understand the effect of simple transformations on the graph of $y = \mathrm{f}(x)$, including sketching associated graphs:** $\boldsymbol{y = a\mathrm{f}(x), \quad y = \mathrm{f}(x) + a,}$ $\boldsymbol{y = \mathrm{f}(x + a), \; y = \mathrm{f}(ax)}$ and combinations of these transformations	Students should be able to find the graphs of $y = \lvert \mathrm{f}(x) \rvert$ and $y = \lvert \mathrm{f}(-x) \rvert$, given the graph of $y = \mathrm{f}(x)$. Students should be able to apply a combination of these transformations to any of the functions in the A Level specification (quadratics, cubics, quartics, reciprocal, $\frac{a}{x^2}$, $\lvert x \rvert$, $\sin x$, $\cos x$, $\tan x$, e^x and a^x) and sketch the resulting graph. Given the graph of $y = \mathrm{f}(x)$, students should be able to sketch the graph of, e.g. $y = 2\mathrm{f}(3x)$, or $y = \mathrm{f}(-x) + 1$, and should be able to sketch (for example) $y = 3 + \sin 2x$, $y = -\cos\left(x + \frac{\pi}{4}\right)$
	2.10	Decompose rational functions into partial fractions (denominators not more complicated than squared linear terms and with no more than 3 terms, numerators constant or linear).	Partial fractions to include denominators such as $(ax + b)(cx + d)(ex + f)$ and $(ax + b)(cx + d)^2$. Applications to integration, differentiation and series expansions.

<table>
<tr><th rowspan="2">Topics</th><th colspan="3">What students need to learn:</th></tr>
<tr><th colspan="2">Content</th><th>Guidance</th></tr>
<tr><td>2

Algebra and functions

continued</td><td>2.11</td><td>Use of functions in modelling, including consideration of limitations and refinements of the models.</td><td>For example, use of trigonometric functions for modelling tides, hours of sunlight, etc. Use of exponential functions for growth and decay (see Paper 1, Section 6.7). Use of reciprocal function for inverse proportion (e.g. pressure and volume).</td></tr>
<tr><td rowspan="2">3

Coordinate geometry in the (x,y) plane</td><td>3.1</td><td>Understand and use the equation of a straight line, including the forms $y - y_1 = m(x - x_1)$ and $ax + by + c = 0$;

Gradient conditions for two straight lines to be parallel or perpendicular.

Be able to use straight line models in a variety of contexts.</td><td>To include the equation of a line through two given points, and the equation of a line parallel (or perpendicular) to a given line through a given point.

$m' = m$ for parallel lines and $m' = -\frac{1}{m}$ for perpendicular lines

For example, the line for converting degrees Celsius to degrees Fahrenheit, distance against time for constant speed, etc.</td></tr>
<tr><td>3.2</td><td>Understand and use the coordinate geometry of the circle including using the equation of a circle in the form $(x - a)^2 + (y - b)^2 = r^2$

Completing the square to find the centre and radius of a circle; use of the following properties:
• the angle in a semicircle is a right angle
• the perpendicular from the centre to a chord bisects the chord
• the radius of a circle at a given point on its circumference is perpendicular to the tangent to the circle at that point.</td><td>Students should be able to find the radius and the coordinates of the centre of the circle given the equation of the circle, and vice versa.

Students should also be familiar with the equation $x^2 + y^2 + 2fx + 2gy + c = 0$

Students should be able to find the equation of a circumcircle of a triangle with given vertices using these properties.

Students should be able to find the equation of a tangent at a specified point, using the perpendicular property of tangent and radius.</td></tr>
</table>

Topics	What students need to learn:		
	Content		Guidance
3 **Coordinate geometry in the (x, y) plane** *continued*	3.3	Understand and use the parametric equations of curves and conversion between Cartesian and parametric forms.	For example: $x = 3\cos t,\ y = 3\sin t$ describes a circle centre O radius 3 $x = 2 + 5\cos t,\ y = -4 + 5\sin t$ describes a circle centre $(2, -4)$ with radius 5 $x = 5t,\ y = \frac{5}{t}$ describes the curve $xy = 25$ (or $y = \frac{25}{x}$) $x = 5t, y = 3t^2$ describes the quadratic curve $25y = 3x^2$ and other familiar curves covered in the specification. Students should pay particular attention to the domain of the parameter t, as a specific section of a curve may be described.
	3.4	Use parametric equations in modelling in a variety of contexts.	A shape may be modelled using parametric equations or students may be asked to find parametric equations for a motion. For example, an object moves with constant velocity from $(1, 8)$ at $t = 0$ to $(6, 20)$ at $t = 5$. This may also be tested in Paper 3, section 7 (kinematics).
4 **Sequences and series**	4.1	**Understand and use the binomial expansion of $(a + bx)^n$ for positive integer n; the notations $n!$ and ^nC_r link to binomial probabilities.**	**Use of Pascal's triangle.** **Relation between binomial coefficients.** **Also be aware of alternative notations such as $\binom{n}{r}$ and ^nC_r** **Considered further in Paper 3 Section 4.1.**
		Extend to any rational n, including its use for approximation; be aware that the expansion is valid for $\left\lvert \frac{bx}{a} \right\rvert < 1$ (proof not required)	May be used with the expansion of rational functions by decomposition into partial fractions May be asked to comment on the range of validity.

Topics	What students need to learn:		
	Content		Guidance
4 **Sequences and series** *continued*	4.2	Work with sequences including those given by a formula for the nth term and those generated by a simple relation of the form $x_{n+1} = \mathrm{f}(x_n)$; increasing sequences; decreasing sequences; periodic sequences.	For example $u_n = \dfrac{1}{3n+1}$ describes a decreasing sequence as $u_{n+1} < u_n$ for all integer n $u_n = 2^n$ is an increasing sequence as $u_{n+1} > u_n$ for all integer n $u_{n+1} = \dfrac{1}{u_n}$ for $n > 1$ and $u_1 = 3$ describes a periodic sequence of order 2
	4.3	Understand and use sigma notation for sums of series.	Knowledge that $\sum_{1}^{n} 1 = n$ is expected
	4.4	Understand and work with arithmetic sequences and series, including the formulae for nth term and the sum to n terms	The proof of the sum formula for an arithmetic sequence should be known including the formula for the sum of the first n natural numbers.
	4.5	Understand and work with geometric sequences and series, including the formulae for the nth term and the sum of a finite geometric series; the sum to infinity of a convergent geometric series, including the use of $\lvert r\rvert < 1$; modulus notation	The proof of the sum formula should be known. Given the sum of a series students should be able to use logs to find the value of n. The sum to infinity may be expressed as S_∞
	4.6	Use sequences and series in modelling.	Examples could include amounts paid into saving schemes, increasing by the same amount (arithmetic) or by the same percentage (geometric) or could include other series defined by a formula or a relation.

Topics	What students need to learn:		
	Content		Guidance
5 **Trigonometry**	5.1	**Understand and use the definitions of sine, cosine and tangent for all arguments;** **the sine and cosine rules;** **the area of a triangle in the form $\frac{1}{2}ab\sin C$** Work with radian measure, including use for arc length and area of sector.	**Use of x and y coordinates of points on the unit circle to give cosine and sine respectively,** **including the ambiguous case of the sine rule.** Use of the formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ for arc lengths and areas of sectors of a circle.
	5.2	Understand and use the standard small angle approximations of sine, cosine and tangent $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$, $\tan\theta \approx \theta$ Where θ is in radians.	Students should be able to approximate, e.g. $\frac{\cos 3x - 1}{x\sin 4x}$ when x is small, to $-\frac{9}{8}$
	5.3	**Understand and use the sine, cosine and tangent functions; their graphs, symmetries and periodicity.** Know and use exact values of sin and cos for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ and multiples thereof, and exact values of tan for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \pi$ and multiples thereof.	**Knowledge of graphs of curves with equations such as $y = \sin x$, $y = \cos(x + 30°)$, $y = \tan 2x$ is expected.**
	5.4	Understand and use the definitions of secant, cosecant and cotangent and of arcsin, arccos and arctan; their relationships to sine, cosine and tangent; understanding of their graphs; their ranges and domains.	Angles measured in both degrees and radians.

Topics	What students need to learn:		
	Content		Guidance
5 Trigonometry *continued*	5.5	**Understand and use** $\tan\theta = \frac{\sin\theta}{\cos\theta}$ **Understand and use** $\sin^2\theta + \cos^2\theta = 1$ $\sec^2\theta = 1 + \tan^2\theta$ and $\text{cosec}^2\theta = 1 + \cot^2\theta$	**These identities may be used to solve trigonometric equations** and angles may be in degrees or radians. **They may also be used to prove further identities.**
	5.6	Understand and use double angle formulae; use of formulae for $\sin(A \pm B)$, $\cos(A \pm B)$, and $\tan(A \pm B)$, understand geometrical proofs of these formulae.	To include application to half angles. Knowledge of the $\tan(\frac{1}{2}\theta)$ formulae will *not* be required.
		Understand and use expressions for $a\cos\theta + b\sin\theta$ in the equivalent forms of $r\cos(\theta \pm \alpha)$ or $r\sin(\theta \pm \alpha)$	Students should be able to solve equations such as $a\cos\theta + b\sin\theta = c$ in a given interval.
	5.7	**Solve simple trigonometric equations in a given interval, including quadratic equations in sin, cos and tan and equations involving multiples of the unknown angle.**	**Students should be able to solve equations such as** $\mathbf{\sin(x + 70°) = 0.5}$ **for** $\mathbf{0 < x < 360°}$, $\mathbf{3 + 5\cos 2x = 1}$ **for** $\mathbf{-180° < x < 180°}$ $\mathbf{6\cos^2 x + \sin x - 5 = 0, 0 \leqslant x < 360°}$ These may be in degrees or radians and this will be specified in the question.
	5.8	Construct proofs involving trigonometric functions and identities.	Students need to prove identities such as $\cos x\cos 2x + \sin x\sin 2x \equiv \cos x$.
	5.9	Use trigonometric functions to solve problems in context, including problems involving vectors, kinematics and forces.	Problems could involve (for example) wave motion, the height of a point on a vertical circular wheel, or the hours of sunlight throughout the year. Angles may be measured in degrees or in radians.

Topics	What students need to learn:		
		Content	**Guidance**
6 **Exponentials and logarithms**	6.1	**Know and use the function a^x and its graph, where a is positive.** **Know and use the function e^x and its graph.**	**Understand the difference in shape between $a < 1$ and $a > 1$** To include the graph of $y = e^{ax+b} + c$
	6.2	**Know that the gradient of e^{kx} is equal to ke^{kx} and hence understand why the exponential model is suitable in many applications.**	**Realise that when the rate of change is proportional to the y value, an exponential model should be used.**
	6.3	**Know and use the definition of $\log_a x$ as the inverse of a^x, where a is positive and $x \geqslant 0$.** **Know and use the function $\ln x$ and its graph.** **Know and use $\ln x$ as the inverse function of e^x**	$a \neq 1$ **Solution of equations of the form $e^{ax+b} = p$ and $\ln(ax + b) = q$ is expected.**
	6.4	**Understand and use the laws of logarithms:** $\log_a x + \log_a y = \log_a (xy)$ $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$ $k \log_a x = \log_a x^k$ **(including, for example, $k = -1$ and $k = -\frac{1}{2}$)**	**Includes $\log_a a = 1$**
	6.5	**Solve equations of the form $a^x = b$**	**Students may use the change of base formula. Questions may be of the form, e.g. $2^{3x-1} = 3$**
	6.6	**Use logarithmic graphs to estimate parameters in relationships of the form $y = ax^n$ and $y = kb^x$, given data for x and y**	**Plot $\log y$ against $\log x$ and obtain a straight line where the intercept is $\log a$ and the gradient is n** **Plot $\log y$ against x and obtain a straight line where the intercept is $\log k$ and the gradient is $\log b$**

Topics	What students need to learn:		
	Content		Guidance
6 **Exponentials and logarithms** *continued*	6.7	**Understand and use exponential growth and decay; use in modelling (examples may include the use of e in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as a model for population growth); consideration of limitations and refinements of exponential models.**	**Students may be asked to find the constants used in a model.** **They need to be familiar with terms such as initial, meaning when $t = 0$.** **They may need to explore the behaviour for large values of t or to consider whether the range of values predicted is appropriate.** **Consideration of an improved model may be required.**
7 **Differentiation**	7.1	**Understand and use the derivative of $\mathrm{f}(x)$ as the gradient of the tangent to the graph of $y = \mathrm{f}(x)$ at a general point (x, y); the gradient of the tangent as a limit; interpretation as a rate of change** **sketching the gradient function for a given curve** **second derivatives** **differentiation from first principles for small positive integer powers of x and for $\sin x$ and $\cos x$**	**Know that $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the rate of change of y with respect to x.** **The notation $\mathrm{f}'(x)$ may be used for the first derivative and $\mathrm{f}''(x)$ may be used for the second derivative.** **Given for example the graph of $y = \mathrm{f}(x)$, sketch the graph of $y = \mathrm{f}'(x)$ using given axes and scale. This could relate speed and acceleration for example.** **For example, students should be able to use, for $n = 2$ and $n = 3$, the gradient expression** $\lim_{h \to 0}\left(\frac{(x+h)^n - x^n}{h}\right)$ **Students may use δx or h**

Topics	What students need to learn:		
	Content		Guidance
7 **Differentiation** *continued*	7.1 *cont.*	**Understand and use the second derivative as the rate of change of gradient;** connection to convex and concave sections of curves and points of inflection.	**Use the condition $\mathrm{f}''(x) > 0$ implies a minimum and $\mathrm{f}''(x) < 0$ implies a maximum for points where $\mathrm{f}'(x) = 0$** Know that at an inflection point $\mathrm{f}''(x)$ changes sign. Consider cases where $\mathrm{f}''(x) = 0$ and $\mathrm{f}'(x) = 0$ where the point may be a minimum, a maximum or a point of inflection (e.g. $y = x^n$, $n > 2$)
	7.2	**Differentiate x^n, for rational values of *n*, and related constant multiples, sums and differences.** Differentiate e^{kx} and a^{kx}, $\sin kx$, $\cos kx$, $\tan kx$ and related sums, differences and constant multiples. Understand and use the derivative of $\ln x$	**For example, the ability to differentiate expressions such as $(2x + 5)(x - 1)$ and $\dfrac{x^2 + 3x - 5}{4x^{\frac{1}{2}}}$, $x > 0$, is expected.** Knowledge and use of the result $\dfrac{\mathrm{d}}{\mathrm{d}x}(a^{kx}) = ka^{kx} \ln a$ is expected.
	7.3	**Apply differentiation to find gradients, tangents and normals** **maxima and minima and stationary points.** points of inflection **Identify where functions are increasing or decreasing.**	**Use of differentiation to find equations of tangents and normals at specific points on a curve.** **To include applications to curve sketching. Maxima and minima problems may be set in the context of a practical problem.** **To include applications to curve sketching.**

Topics	What students need to learn:		
	Content		Guidance
7 **Differentiation** *continued*	7.4	Differentiate using the product rule, the quotient rule and the chain rule, including problems involving connected rates of change and inverse functions.	Differentiation of $\operatorname{cosec} x$, $\cot x$ and $\sec x$. Differentiation of functions of the form $x = \sin y$, $x = 3\tan 2y$ and the use of $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)}$ Use of connected rates of change in models, e.g. $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$ Skill will be expected in the differentiation of functions generated from standard forms using products, quotients and composition, such as $2x^4 \sin x$, $\frac{e^{3x}}{x}$, $\cos^2 x$ and $\tan^2 2x$.
	7.5	Differentiate simple functions and relations defined implicitly or parametrically, for first derivative only.	The finding of equations of tangents and normals to curves given parametrically or implicitly is required.
	7.6	Construct simple differential equations in pure mathematics and in context, (contexts may include kinematics, population growth and modelling the relationship between price and demand).	Set up a differential equation using given information. For example: In a simple model, the rate of decrease of the radius of the mint is inversely proportional to the square of the radius.
8 **Integration**	8.1	**Know and use the Fundamental Theorem of Calculus**	**Integration as the reverse process of differentiation. Students should know that for indefinite integrals a constant of integration is required.**
	8.2	**Integrate x^n (excluding $n = -1$) and related sums, differences and constant multiples.** Integrate e^{kx}, $\frac{1}{x}$, $\sin kx$, $\cos kx$ and related sums, differences and constant multiples.	**For example, the ability to integrate expressions such as $\frac{1}{2}x^2 - 3x^{-\frac{1}{2}}$ and $\frac{(x+2)^2}{x^{\frac{1}{2}}}$ is expected.** **Given $f'(x)$ and a point on the curve, students should be able to find an equation of the curve in the form $y = f(x)$.** To include integration of standard functions such as $\sin 3x$, $\sec^2 2x$, $\tan x$, e^{5x}, $\frac{1}{2x}$. Students are expected to be able to use trigonometric identities to integrate, for example, $\sin^2 x$, $\tan^2 x$, $\cos^2 3x$.

Topics	What students need to learn:		
	Content		Guidance
8 **Integration** *continued*	8.3	**Evaluate definite integrals; use a definite integral to find the area under a curve** and the area between two curves	**Students will be expected to be able to evaluate the area of a region bounded by a curve and given straight lines,** or between two curves. This includes curves defined parametrically. **For example, find the finite area bounded by the curve $y = 6x - x^2$ and the line $y = 2x$** Or find the finite area bounded by the curve $y = x^2 - 5x + 6$ and the curve $y = 4 - x^2$.
	8.4	Understand and use integration as the limit of a sum.	Recognise $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \lim_{\delta x \to 0} \sum_{x=a}^{b} \mathrm{f}(x)\,\delta x$
	8.5	Carry out simple cases of integration by substitution and integration by parts; understand these methods as the inverse processes of the chain and product rules respectively (Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated; integration by parts includes more than one application of the method but excludes reduction formulae.)	Students should recognise integrals of the form $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\mathrm{d}x = \ln \mathrm{f}(x) + c$. The integral $\int \ln x \,\mathrm{d}x$ is required
	8.6	Integrate using partial fractions that are linear in the denominator.	Integration of rational expressions such as those arising from partial fractions, e.g. $\frac{2}{3x+5}$ Note that the integration of other rational expressions, such as $\frac{x}{x^2+5}$ and $\frac{2}{(2x-1)^4}$ is also required (see previous paragraph).

Topics	What students need to learn:		
	Content		Guidance
8 **Integration** *continued*	8.7	Evaluate the analytical solution of simple first order differential equations with separable variables, including finding particular solutions (Separation of variables may require factorisation involving a common factor.)	Students may be asked to sketch members of the family of solution curves.
	8.8	Interpret the solution of a differential equation in the context of solving a problem, including identifying limitations of the solution; includes links to kinematics.	The validity of the solution for large values should be considered.
9 **Numerical methods**	9.1	Locate roots of $\mathrm{f}(x) = 0$ by considering changes of sign of $\mathrm{f}(x)$ in an interval of x on which $\mathrm{f}(x)$ is sufficiently well behaved. Understand how change of sign methods can fail.	Students should know that sign change is appropriate for continuous functions in a small interval. When the interval is too large sign may not change as there may be an even number of roots. If the function is not continuous, sign may change but there may be an asymptote (not a root).
	9.2	Solve equations approximately using simple iterative methods; be able to draw associated cobweb and staircase diagrams.	Understand that many mathematical problems cannot be solved analytically, but numerical methods permit solution to a required level of accuracy. Use an iteration of the form $x_{n+1} = \mathrm{f}(x_n)$ to find a root of the equation $x = \mathrm{f}(x)$ and show understanding of the convergence in geometrical terms by drawing cobweb and staircase diagrams.
	9.3	Solve equations using the Newton-Raphson method and other recurrence relations of the form $x_{n+1} = \mathrm{g}(x_n)$ Understand how such methods can fail.	For the Newton-Raphson method, students should understand its working in geometrical terms, so that they understand its failure near to points where the gradient is small.

Topics	What students need to learn:		
	Content		Guidance
9 **Numerical methods** *continued*	9.4	Understand and use numerical integration of functions, including the use of the trapezium rule and estimating the approximate area under a curve and limits that it must lie between.	For example, evaluate $\int_0^1 \sqrt{(2x+1)}\ \mathrm{d}x$ using the values of $\sqrt{(2x+1)}$ at $x = 0$, 0.25, 0.5, 0.75 and 1 and use a sketch on a given graph to determine whether the trapezium rule gives an over-estimate or an under-estimate.
	9.5	Use numerical methods to solve problems in context.	Iterations may be suggested for the solution of equations not soluble by analytic means.
10 **Vectors**	10.1	**Use vectors in two dimensions** and in three dimensions	**Students should be familiar with column vectors and with the use of i and j unit vectors in two dimensions** and $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ unit vectors in three dimensions.
	10.2	**Calculate the magnitude and direction of a vector and convert between component form and magnitude/direction form.**	**Students should be able to find a unit vector in the direction of $\mathbf{a}$, and be familiar with the notation $\lvert\mathbf{a}\rvert$.**
	10.3	**Add vectors diagrammatically and perform the algebraic operations of vector addition and multiplication by scalars, and understand their geometrical interpretations.**	**The triangle and parallelogram laws of addition.** **Parallel vectors.**
	10.4	**Understand and use position vectors; calculate the distance between two points represented by position vectors.**	$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ **The distance d between two points (x_1, y_1) and (x_2, y_2) is given by** $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ In three dimensions, the distance d between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$

Topics	What students need to learn:		
	Content		Guidance
10 **Vectors** *continued*	10.5	**Use vectors to solve problems in pure mathematics and in context (including forces).**	**For example, finding position vector of the fourth corner of a shape (e.g. parallelogram) $ABCD$ with three given position vectors for the corners A, B and C.** **Contexts such as velocity, displacement, kinematics and forces will be covered in Paper 3, Sections 6.1, 7.3 and 8.1 – 8.4**

Assessment information

- First assessment: May/June 2018.
- The assessments are 2 hours each.
- The assessments are out of 100 marks.
- Students must answer all questions.
- Calculators can be used in the assessments.
- The booklet *Mathematical Formulae and Statistical Tables* will be provided for use in the assessments.

Synoptic assessment

Synoptic assessment requires students to work across different parts of a qualification and to show their accumulated knowledge and understanding of a topic or subject area.

Synoptic assessment enables students to show their ability to combine their skills, knowledge and understanding with breadth and depth of the subject.

These papers assess synopticity.

Sample assessment materials

A sample paper and mark scheme for these papers can be found in the *Pearson Edexcel Level 3 Advanced GCE in Mathematics Sample Assessment Materials (SAMs)* document.

Paper 3: Statistics and Mechanics

All the Pure Mathematics content is assumed knowledge for Paper 3 and may be tested in parts of questions.

To support the co-teaching of this qualification with the AS Mathematics qualification, common content has been highlighted in bold.

Topics	What students need to learn:		
	Content		Guidance
1 **Statistical sampling**	1.1	**Understand and use the terms 'population' and 'sample'.** **Use samples to make informal inferences about the population.**	**Students will be expected to comment on the advantages and disadvantages associated with a census and a sample.**
		Understand and use sampling techniques, including simple random sampling and opportunity sampling.	**Students will be expected to be familiar with: simple random sampling, stratified sampling, systematic sampling, quota sampling and opportunity (or convenience) sampling.**
		Select or critique sampling techniques in the context of solving a statistical problem, including understanding that different samples can lead to different conclusions about the population.	
2 **Data presentation and interpretation**	2.1	**Interpret diagrams for single-variable data, including understanding that area in a histogram represents frequency.** **Connect to probability distributions.**	**Students should be familiar with histograms, frequency polygons, box and whisker plots (including outliers) and cumulative frequency diagrams.**

Topics	What students need to learn:		
	Content		Guidance
2 **Data presentation and interpretation** *continued*	2.2	**Interpret scatter diagrams and regression lines for bivariate data, including recognition of scatter diagrams which include distinct sections of the population (calculations involving regression lines are excluded).**	**Students should be familiar with the terms explanatory (independent) and response (dependent) variables.** **Use of interpolation and the dangers of extrapolation. Variables other than x and y may be used.** Use to make predictions within the range of values of the explanatory variable. Change of variable may be required, e.g. using knowledge of logarithms to reduce a relationship of the form $y = ax^n$ or $y = kb^x$ into linear form to estimate a and n or k and b.
		Understand informal interpretation of correlation. **Understand that correlation does not imply causation.**	**Use of terms such as positive, negative, zero, strong and weak are expected.**
	2.3	**Interpret measures of central tendency and variation, extending to standard deviation.**	**Data may be discrete, continuous, grouped or ungrouped. Understanding and use of coding.** **Measures of central tendency: mean, median, mode.** **Measures of variation: variance, standard deviation, range and interpercentile ranges.** **Use of linear interpolation to calculate percentiles from grouped data is expected.**
		Be able to calculate standard deviation, including from summary statistics.	**Students should be able to use the statistic x** $S_{xx} = \sum (x - \bar{x})^2 = \sum x^2 - \frac{(\sum x)^2}{n}$ **Use of standard deviation = $\sqrt{\frac{S_{xx}}{n}}$ (or equivalent) is expected but the use of $S = \sqrt{\frac{S_{xx}}{n-1}}$ (as used on spreadsheets) will be accepted.**

Topics	What students need to learn:		
		Content	**Guidance**
2 **Data presentation and interpretation** *continued*	2.4	**Recognise and interpret possible outliers in data sets and statistical diagrams.**	**Any rule needed to identify outliers will be specified in the question.** **For example, use of $Q_1 - 1.5 \times$ IQR and $Q_3 + 1.5 \times$ IQR or mean $\pm 3 \times$ standard deviation.**
		Select or critique data presentation techniques in the context of a statistical problem.	**Students will be expected to draw simple inferences and give interpretations to measures of central tendency and variation.** Significance tests, other than those mentioned in Section 5, will not be expected.
		Be able to clean data, including dealing with missing data, errors and outliers.	**For example, students may be asked to identify possible outliers on a box plot or scatter diagram.**
3 **Probability**	3.1	**Understand and use mutually exclusive and independent events when calculating probabilities.**	**Venn diagrams or tree diagrams may be used.** Set notation to describe events may be used. Use of $\mathrm{P}(B\vert A) = \mathrm{P}(B)$, $\mathrm{P}(A\vert B) = \mathrm{P}(A)$, $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$ in connection with independent events.
		Link to discrete and continuous distributions.	**No formal knowledge of probability density functions is required but students should understand that area under the curve represents probability in the case of a continuous distribution.**
	3.2	Understand and use conditional probability, including the use of tree diagrams, Venn diagrams, two-way tables. Understand and use the conditional probability formula $\mathrm{P}(A\vert B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$	Understanding and use of $\mathrm{P}(A') = 1 - \mathrm{P}(A)$, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$, $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B\vert A)$.

Topics	What students need to learn:		
	Content		Guidance
3 **Probability** *continued*	3.3	Modelling with probability, including critiquing assumptions made and the likely effect of more realistic assumptions.	For example, questioning the assumption that a die or coin is fair.
4 **Statistical distributions**	4.1	**Understand and use simple, discrete probability distributions (calculation of mean and variance of discrete random variables is excluded), including the binomial distribution, as a model; calculate probabilities using the binomial distribution.**	**Students will be expected to use distributions to model a real-world situation and to comment critically on the appropriateness.** **Students should know and be able to identify the discrete uniform distribution.** **The notation $X \sim \mathrm{B}(n, p)$ may be used.** **Use of a calculator to find individual or cumulative binomial probabilities.**
	4.2	Understand and use the Normal distribution as a model; find probabilities using the Normal distribution	The notation $X \sim \mathrm{N}(\mu, \sigma^2)$ may be used. Knowledge of the shape and the symmetry of the distribution is required. Knowledge of the probability density function is not required. Derivation of the mean, variance and cumulative distribution function is not required. Questions may involve the solution of simultaneous equations. Students will be expected to use their calculator to find probabilities connected with the normal distribution.
		Link to histograms, mean, standard deviation, points of inflection	Students should know that the points of inflection on the normal curve are at $x = \mu \pm \sigma$. The derivation of this result is not expected.
		and the binomial distribution.	Students should know that when n is large and p is close to 0.5 the distribution $\mathrm{B}(n, p)$ can be approximated by $\mathrm{N}(np, np[1 - p])$ The application of a continuity correction is expected.

Topics	What students need to learn:		
	Content		Guidance
4 **Statistical distributions** *continued*	4.3	Select an appropriate probability distribution for a context, with appropriate reasoning, including recognising when the binomial or Normal model may not be appropriate.	Students should know under what conditions a binomial distribution or a Normal distribution might be a suitable model.
5 **Statistical hypothesis testing**	5.1	**Understand and apply the language of statistical hypothesis testing, developed through a binomial model: null hypothesis, alternative hypothesis, significance level, test statistic, 1-tail test, 2-tail test, critical value, critical region, acceptance region, *p*-value;** extend to correlation coefficients as measures of how close data points lie to a straight line. and be able to interpret a given correlation coefficient using a given *p*-value or critical value (calculation of correlation coefficients is excluded).	**An informal appreciation that the expected value of a binomial distribution is given by np may be required for a 2-tail test.** Students should know that the product moment correlation coefficient r satisfies $\|r\| \le 1$ and that a value of $r = \pm 1$ means the data points all lie on a straight line. Students will be expected to calculate a value of r using their calculator but use of the formula is not required. Hypotheses should be stated in terms of ρ with a null hypothesis of $\rho = 0$ where ρ represents the population correlation coefficient. Tables of critical values or a *p*-value will be given.

Topics	What students need to learn: Content		Guidance
5 **Statistical hypothesis testing** *continued*	5.2	**Conduct a statistical hypothesis test for the proportion in the binomial distribution and interpret the results in context.**	
		Understand that a sample is being used to make an inference about the population **and**	**Hypotheses should be expressed in terms of the population parameter *p***
		appreciate that the significance level is the probability of incorrectly rejecting the null hypothesis.	**A formal understanding of Type I errors is not expected.**
	5.3	Conduct a statistical hypothesis test for the mean of a Normal distribution with known, given or assumed variance and interpret the results in context.	Students should know that: If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ and that a test for μ can be carried out using: $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1^2)$. No proofs required. Hypotheses should be stated in terms of the population mean μ. Knowledge of the Central Limit Theorem or other large sample approximations is not required.
6 **Quantities and units in mechanics**	6.1	**Understand and use fundamental quantities and units in the S.I. system: length, time, mass.**	
		Understand and use derived quantities and units: velocity, acceleration, force, weight, moment**.**	**Students may be required to convert one unit into another e.g. km h^{-1} into m s^{-1}**

<table>
<tr><th rowspan="2">Topics</th><th colspan="3">What students need to learn:</th></tr>
<tr><th colspan="2">Content</th><th>Guidance</th></tr>
<tr><td rowspan="5">7

Kinematics</td><td>7.1</td><td>Understand and use the language of kinematics: position; displacement; distance travelled; velocity; speed; acceleration.</td><td>Students should know that distance and speed must be positive.</td></tr>
<tr><td>7.2</td><td>Understand, use and interpret graphs in kinematics for motion in a straight line: displacement against time and interpretation of gradient; velocity against time and interpretation of gradient and area under the graph.</td><td>Graphical solutions to problems may be required.</td></tr>
<tr><td>7.3</td><td>Understand, use and derive the formulae for constant acceleration for motion in a straight line.

Extend to 2 dimensions using vectors.</td><td>Derivation may use knowledge of sections 7.2 and/or 7.4

Understand and use suvat formulae for constant acceleration in 2-D,

e.g. $\mathbf{v} = \mathbf{u} + \mathbf{a}t$, $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ with vectors given in $\mathbf{i} - \mathbf{j}$ or column vector form.

Use vectors to solve problems.</td></tr>
<tr><td>7.4</td><td>Use calculus in kinematics for motion in a straight line:

$v = \frac{\mathrm{d}r}{\mathrm{d}t}$, $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2}$

$r = \int v \ \mathrm{d}t$, $v = \int a \ \mathrm{d}t$

Extend to 2 dimensions using vectors.</td><td>The level of calculus required will be consistent with that in Sections 7 and 8 in the Pure Mathematics content.

Differentiation and integration of a vector with respect to time. e.g.

Given $\mathbf{r} = t^2\mathbf{i} + t^{\frac{3}{2}}\mathbf{j}$, find $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ at a given time.</td></tr>
<tr><td>7.5</td><td>Model motion under gravity in a vertical plane using vectors; projectiles.</td><td>Derivation of formulae for time of flight, range and greatest height and the derivation of the equation of the path of a projectile may be required.</td></tr>
</table>

Topics	What students need to learn:		
	Content		Guidance
8 **Forces and Newton's laws**	8.1	**Understand the concept of a force; understand and use Newton's first law.**	**Normal reaction, tension, thrust or compression, resistance.**
	8.2	**Understand and use Newton's second law for motion in a straight line (restricted to forces in two perpendicular directions or simple cases of forces given as 2-D vectors);** extend to situations where forces need to be resolved (restricted to 2 dimensions).	**Problems will involve motion in a straight line with constant acceleration in scalar form, where the forces act either parallel or perpendicular to the motion.** **Problems may involve motion in a straight line with constant acceleration in vector form, where the forces are given in $\mathbf{i} - \mathbf{j}$ form or as column vectors.** Extend to problems where forces need to be resolved, e.g. a particle moving on an inclined plane.
	8.3	**Understand and use weight and motion in a straight line under gravity; gravitational acceleration, g, and its value in S.I. units to varying degrees of accuracy.** **(The inverse square law for gravitation is not required and g may be assumed to be constant, but students should be aware that g is not a universal constant but depends on location.)**	**The default value of g will be 9.8 m s^{-2} but some questions may specify another value, e.g. g = 10 m s^{-2}**

Topics	What students need to learn:		
	Content		Guidance
8 **Forces and Newton's laws** *continued*	8.4	**Understand and use Newton's third law; equilibrium of forces on a particle and motion in a straight line (restricted to forces in two perpendicular directions or simple cases of forces given as 2-D vectors); application to problems involving smooth pulleys and connected particles;** resolving forces in 2 dimensions; equilibrium of a particle under coplanar forces.	**Connected particle problems could include problems with particles in contact e.g. lift problems.** Problems may be set where forces need to be resolved, e.g. at least one of the particles is moving on an inclined plane.
	8.5	Understand and use addition of forces; resultant forces; dynamics for motion in a plane.	Students may be required to resolve a vector into two components or use a vector diagram, e.g. problems involving two or more forces, given in magnitude-direction form.
	8.6	Understand and use the $\mathrm{F} \le \mu R$ model for friction; coefficient of friction; motion of a body on a rough surface; limiting friction and statics.	An understanding of $\mathrm{F} = \mu R$ when a particle is moving. An understanding of $\mathrm{F} \le \mu R$ in a situation of equilibrium.
9 **Moments**	9.1	Understand and use moments in simple static contexts.	Equilibrium of rigid bodies. Problems involving parallel and non-parallel coplanar forces, e.g. ladder problems.

Assessment information

- First assessment: May/June 2018.
- The assessment is 2 hours.
- The assessment is out of 100 marks.
- Students must answer all questions.
- Calculators can be used in the assessment.
- The booklet '*Mathematical Formulae and Statistical Tables'* will be provided for use in the assessment.

Synoptic assessment

Synoptic assessment requires students to work across different parts of a qualification and to show their accumulated knowledge and understanding of a topic or subject area.

Synoptic assessment enables students to show their ability to combine their skills, knowledge and understanding with breadth and depth of the subject.

This paper assesses synopticity.

Sample assessment materials

A sample paper and mark scheme for this paper can be found in the *Pearson Edexcel Level 3 Advanced GCE in Mathematics Sample Assessment Materials (SAMs)* document.

Assessment Objectives

Students must:		% in GCE A Level
AO1	**Use and apply standard techniques** Students should be able to: • select and correctly carry out routine procedures; and • accurately recall facts, terminology and definitions	48–52
AO2	**Reason, interpret and communicate mathematically** Students should be able to: • construct rigorous mathematical arguments (including proofs) • make deductions and inferences • assess the validity of mathematical arguments • explain their reasoning; and • use mathematical language and notation correctly. *Where questions/tasks targeting this Assessment Objective will also credit candidates for the ability to 'use and apply standard techniques' (AO1) and/or to 'solve problems within mathematics and in other contexts' (AO3) an appropriate proportion of the marks for the question/task must be attributed to the corresponding Assessment Objective(s).*	23–27
AO3	**Solve problems within mathematics and in other contexts** Students should be able to: • translate problems in mathematical and non-mathematical contexts into mathematical processes • interpret solutions to problems in their original context, and, where appropriate, evaluate their accuracy and limitations • translate situations in context into mathematical models • use mathematical models; and • evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them. *Where questions/tasks targeting this Assessment Objective will also credit candidates for the ability to 'use and apply standard techniques' (AO1) and/or to 'reason, interpret and communicate mathematically' (AO2) an appropriate proportion of the marks for the question/task must be attributed to the corresponding Assessment Objective(s).*	23–27
	Total	**100%**

Further guidance on the interpretation of these assessment objectives is given in *Appendix 4*.

Breakdown of Assessment Objectives

Paper	Assessment Objectives			Total for all Assessment Objectives
	AO1 %	AO2 %	AO3 %	
Paper 1: Pure Mathematics 1	16.00–17.33	8.66–10	6.33–7.67	33.33%
Paper 2: Pure Mathematics 2	16.00–17.33	8.66–10	6.33–7.67	33.33%
Paper 3: Statistics and Mechanics	16.00–17.33	5.66–7.00	10.33–11.67	33.33%
Total for GCE A Level	**48-52**	**23-27**	**23-27**	**100%**

NB: Totals have been rounded either up or down.

3 Administration and general information

Entries

Details of how to enter students for the examinations for this qualification can be found in our *UK Information Manual*. A copy is made available to all examinations officers and is available on our website: qualifications.pearson.com

Discount code and performance tables

Centres should be aware that students who enter for more than one GCE qualification with the same discount code will have only one of the grades they achieve counted for the purpose of the school and college performance tables. This will be the grade for the larger qualification (i.e. the A Level grade rather than the AS grade). If the qualifications are the same size, then the better grade will be counted (please see *Appendix 8*: *Codes*).

Students should be advised that if they take two GCE qualifications with the same discount code, colleges, universities and employers which they wish to progress to are likely to take the view that this achievement is equivalent to only one GCE. The same view may be taken if students take two GCE qualifications that have different discount codes but have significant overlap of content. Students or their advisers who have any doubts about their subject combinations should check with the institution they wish to progress to before embarking on their programmes.

Access arrangements, reasonable adjustments, special consideration and malpractice

Equality and fairness are central to our work. Our equality policy requires all students to have equal opportunity to access our qualifications and assessments, and our qualifications to be awarded in a way that is fair to every student.

We are committed to making sure that:

- students with a protected characteristic (as defined by the Equality Act 2010) are not, when they are undertaking one of our qualifications, disadvantaged in comparison to students who do not share that characteristic
- all students achieve the recognition they deserve for undertaking a qualification and that this achievement can be compared fairly to the achievement of their peers.

Language of assessment

Assessment of this qualification will be available in English. All student work must be in English.

Access arrangements

Access arrangements are agreed before an assessment. They allow students with special educational needs, disabilities or temporary injuries to:

- access the assessment
- show what they know and can do without changing the demands of the assessment.

The intention behind an access arrangement is to meet the particular needs of an individual student with a disability, without affecting the integrity of the assessment. Access arrangements are the principal way in which awarding bodies comply with the duty under the Equality Act 2010 to make 'reasonable adjustments'.

Access arrangements should always be processed at the start of the course. Students will then know what is available and have the access arrangement(s) in place for assessment.

Reasonable adjustments

The Equality Act 2010 requires an awarding organisation to make reasonable adjustments where a person with a disability would be at a substantial disadvantage in undertaking an assessment. The awarding organisation is required to take reasonable steps to overcome that disadvantage.

A reasonable adjustment for a particular person may be unique to that individual and therefore might not be in the list of available access arrangements.

Whether an adjustment will be considered reasonable will depend on a number of factors, including:

- the needs of the student with the disability
- the effectiveness of the adjustment
- the cost of the adjustment; and
- the likely impact of the adjustment on the student with the disability and other students.

An adjustment will not be approved if it involves unreasonable costs to the awarding organisation, or affects timeframes or the security or integrity of the assessment. This is because the adjustment is not 'reasonable'.

Special consideration

Special consideration is a post-examination adjustment to a student's mark or grade to reflect temporary injury, illness or other indisposition at the time of the examination/ assessment, which has had, or is reasonably likely to have had, a material effect on a candidate's ability to take an assessment or demonstrate their level of attainment in an assessment.

Further information

Please see our website for further information about how to apply for access arrangements and special consideration.

For further information about access arrangements, reasonable adjustments and special consideration, please refer to the JCQ website: www.jcq.org.uk.

Malpractice

Candidate malpractice

Candidate malpractice refers to any act by a candidate that compromises or seeks to compromise the process of assessment or which undermines the integrity of the qualifications or the validity of results/certificates.

Candidate malpractice in examinations **must** be reported to Pearson using a *JCQ Form M1* (available at www.jcq.org.uk/exams-office/malpractice). The form should be emailed to candidatemalpractice@pearson.com. Please provide as much information and supporting documentation as possible. Note that the final decision regarding appropriate sanctions lies with Pearson.

Failure to report malpractice constitutes staff or centre malpractice.

Staff/centre malpractice

Staff and centre malpractice includes both deliberate malpractice and maladministration of our qualifications. As with candidate malpractice, staff and centre malpractice is any act that compromises or seeks to compromise the process of assessment or which undermines the integrity of the qualifications or the validity of results/certificates.

All cases of suspected staff malpractice and maladministration **must** be reported immediately, before any investigation is undertaken by the centre, to Pearson on a *JCQ Form M2(a)* (available at www.jcq.org.uk/exams-office/malpractice). The form, supporting documentation and as much information as possible should be emailed to pqsmalpractice@pearson.com. Note that the final decision regarding appropriate sanctions lies with Pearson.

Failure to report malpractice itself constitutes malpractice.

More detailed guidance on malpractice can be found in the latest version of the document *General and Vocational Qualifications Suspected Malpractice in Examinations and Assessments Policies and Procedures,* available at www.jcq.org.uk/exams-office/malpractice.

Awarding and reporting

This qualification will be graded, awarded and certificated to comply with the requirements of Ofqual's General Conditions of Recognition.

This A Level qualification will be graded and certificated on a six-grade scale from A* to E using the total combined marks (out of 300) for the three compulsory papers. Individual papers are not graded.

Students whose level of achievement is below the minimum judged by Pearson to be of sufficient standard to be recorded on a certificate will receive an unclassified U result.

The first certification opportunity for this qualification will be 2018.

Student recruitment and progression

Pearson follows the JCQ policy concerning recruitment to our qualifications in that:

- they must be available to anyone who is capable of reaching the required standard
- they must be free from barriers that restrict access and progression
- equal opportunities exist for all students.

Prior learning and other requirements

There are no prior learning or other requirements for this qualification.

Students who would benefit most from studying this qualification are likely to have a Level 2 qualification such as a GCSE in Mathematics.

Progression

Students can progress from this qualification to:

- a range of different, relevant academics or vocational higher education qualifications
- employment in a relevant sector
- further training.

Appendices

Appendix 1: Formulae

Formulae that students are expected to know for A Level Mathematics are given below and will not appear in the booklet *Mathematical Formulae and Statistical Tables*, which will be provided for use with the paper.

Pure Mathematics

Quadratic Equations

$ax^2 + bx + c = 0$ has roots $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Laws of Indices

$a^x a^y \equiv a^{x+y}$

$a^x \div a^y \equiv a^{x-y}$

$(a^x)^y \equiv a^{xy}$

Laws of Logarithms

$x = a^n \Leftrightarrow n = \log_a x$ for $a > 0$ and $x > 0$

$\log_a x + \log_a y \equiv \log_a (xy)$

$\log_a x - \log_a y \equiv \log_a \left(\dfrac{x}{y}\right)$

$k \log_a x \equiv \log_a \left(x^k\right)$

Coordinate Geometry

A straight line graph, gradient m passing through (x_1, y_1) has equation $y - y_1 = m(x - x_1)$

Straight lines with gradients m_1 and m_2 are perpendicular when $m_1 m_2 = -1$

Sequences

General term of an arithmetic progression:

$u_n = a + (n-1)d$

General term of a geometric progression:

$u_n = ar^{n-1}$

Trigonometry

In the triangle ABC

Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$

Area $= \dfrac{1}{2}ab\sin C$

$\cos^2 A + \sin^2 A \equiv 1$

$\sec^2 A \equiv 1 + \tan^2 A$

$\operatorname{cosec}^2 A \equiv 1 + \cot^2 A$

$\sin 2A \equiv 2 \sin A \cos A$

$\cos 2A \equiv \cos^2 A - \sin^2 A$

$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$$

Mensuration

Circumference and area of circle, radius r and diameter d:

$C = 2\pi r = \pi d \qquad A = \pi r^2$

Pythagoras' theorem:

In any right-angled triangle where a, b and c are the lengths of the sides and c is the hypotenuse, $c^2 = a^2 + b^2$

Area of a trapezium = $\dfrac{1}{2}(a+b)h$, where a and b are the lengths of the parallel sides and h is their perpendicular separation.

Volume of a prism = area of cross section × length

For a circle of radius r, where an angle at the centre of θ radians subtends an arc of length s and encloses an associated sector of area A:

$$s = r\theta \qquad A = \frac{1}{2}r^2\theta$$

Calculus and Differential Equations

Differentiation

Function	Derivative
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
e^{kx}	ke^{kx}
$\ln x$	$\frac{1}{x}$
$f(x)+g(x)$	$f'(x)+g'(x)$
$f(x)g(x)$	$f'(x)g(x)+f(x)g'(x)$
$f(g(x))$	$f'(g(x))g'(x)$

Integration

Function	Integral
x^n	$\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$
$\cos kx$	$\frac{1}{k}\sin kx + c$
$\sin kx$	$-\frac{1}{k}\cos kx + c$
e^{kx}	$\frac{1}{k}e^{kx}+c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c,\ x\neq 0$
$f'(x)+g'(x)$	$f(x)+g(x)+c$
$f'(g(x))g'(x)$	$f(g(x))+c$

Area under a curve $= \int_a^b y\ \mathrm{d}x\ (y \geqslant 0)$

Vectors

$$|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{(x^2 + y^2 + z^2)}$$

Statistics

The mean of a set of data: $\bar{x} = \dfrac{\sum x}{n} = \dfrac{\sum \mathrm{f}x}{\sum \mathrm{f}}$

The standard Normal variable: $Z = \dfrac{X - \mu}{\sigma}$ where $X \sim \mathrm{N}(\mu, \sigma^2)$

Mechanics

Forces and Equilibrium

Weight = mass × g

Friction: $F \leqslant \mu R$

Newton's second law in the form: $F = ma$

Kinematics

For motion in a straight line with variable acceleration:

$$v = \frac{\mathrm{d}r}{\mathrm{d}t} \qquad a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2}$$

$$r = \int v \,\mathrm{d}t \qquad v = \int a \,\mathrm{d}t$$

Appendix 2: Notation

The tables below set out the notation that must be used in A Level Mathematics examinations. Students will be expected to understand this notation without need for further explanation.

1	Set notation	
1.1	$\in$	is an element of
1.2	$\notin$	is not an element of
1.3	$\subseteq$	is a subset of
1.4	$\subset$	is a proper subset of
1.5	$\{x_1, x_2, \ldots\}$	the set with elements $x_1, x_2, \ldots$
1.6	$\{x : \ldots\}$	the set of all x such that $\ldots$
1.7	$\mathrm{n}(A)$	the number of elements in set A
1.8	$\varnothing$	the empty set
1.9	ε	the universal set
1.10	A'	the complement of the set A
1.11	$\mathbb{N}$	the set of natural numbers, $\{1, 2, 3, \ldots\}$
1.12	$\mathbb{Z}$	the set of integers, $\{0, \pm1, \pm2, \pm3, \ldots\}$
1.13	$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, \ldots\}$
1.14	$\mathbb{Z}_0^+$	the set of non-negative integers, $\{0, 1, 2, 3, \ldots\}$
1.15	$\mathbb{R}$	the set of real numbers
1.16	$\mathbb{Q}$	the set of rational numbers, $\left\{\frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{Z}^+\right\}$
1.17	$\cup$	union
1.18	$\cap$	intersection
1.19	(x, y)	the ordered pair x, y
1.20	$[a, b]$	the closed interval $\{x \in \mathbb{R} : a \le x \le b\}$
1.21	$[a, b)$	the interval $\{x \in \mathbb{R} : a \le x < b\}$
1.22	$(a, b]$	the interval $\{\{x \in \mathbb{R} : a < x \le b\}$
1.23	(a, b)	the open interval $\{x \in \mathbb{R} : a < x < b\}$

2	Miscellaneous symbols	
2.1	$=$	is equal to
2.2	$\neq$	is not equal to
2.3	$\equiv$	is identical to or is congruent to
2.4	$\approx$	is approximately equal to
2.5	∞	infinity
2.6	$\propto$	is proportional to
2.7	$\therefore$	therefore
2.8	$\because$	because
2.9	$<$	is less than
2.10	$\leqslant$, $\leq$	is less than or equal to, is not greater than
2.11	$>$	is greater than
2.12	$\geqslant$, $\geq$	is greater than or equal to, is not less than
2.13	$p \Rightarrow q$	p implies q (if p then q)
2.14	$p \Leftarrow q$	p is implied by q (if q then p)
2.15	$p \Leftrightarrow q$	p implies and is implied by q (p is equivalent to q)
2.16	a	first term for an arithmetic or geometric sequence
2.17	l	last term for an arithmetic sequence
2.18	d	common difference for an arithmetic sequence
2.19	r	common ratio for a geometric sequence
2.20	S_n	sum to n terms of a sequence
2.21	S_∞	sum to infinity of a sequence

3	Operations	
3.1	$a + b$	a plus b
3.2	$a - b$	a minus b
3.3	$a \times b,\ ab,\ a \cdot b$	a multiplied by b
3.4	$a \div b,\ \frac{a}{b}$	a divided by b
3.5	$\sum_{i=1}^{n} a_i$	$a_1 + a_2 + \ldots + a_n$
3.6	$\prod_{i=1}^{n} a_i$	$a_1 \times a_2 \times \ldots \times a_n$
3.7	$\sqrt{a}$	the non-negative square root of a
3.8	$\lvert a \rvert$	the modulus of a
3.9	$n!$	n factorial: $n! = n \times (n-1) \times \ldots \times 2 \times 1$, $n \in \mathbb{N}$; $0! = 1$
3.10	$\binom{n}{r},\ {}^nC_r,\ {}_nC_r$	the binomial coefficient $\frac{n!}{r!(n-r)!}$ for $n, r \in \mathbb{Z}_0^+, r \leqslant n$ or $\frac{n(n-1)\ldots(n-r+1)}{r!}$ for $n \in \mathbb{Q}, r \in \mathbb{Z}_0^+$

4	Functions	
4.1	$\mathrm{f}(x)$	the value of the function f at x
4.2	$\mathrm{f}: x \mapsto y$	the function f maps the element x to the element y
4.3	f^{-1}	the inverse function of the function f
4.4	gf	the composite function of f and g which is defined by $\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x))$
4.5	$\lim_{x \to a} \mathrm{f}(x)$	the limit of $\mathrm{f}(x)$ as x tends to a
4.6	$\Delta x,\ \delta x$	an increment of x
4.7	$\frac{\mathrm{d}y}{\mathrm{d}x}$	the derivative of y with respect to x
4.8	$\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$	the nth derivative of y with respect to x
4.9	$\mathrm{f}'(x),\ \mathrm{f}''(x),\ \ldots,\ \mathrm{f}^{(n)}(x)$	the first, second, ..., n^{th} derivatives of $\mathrm{f}(x)$ with respect to x

4	Functions	
4.10	$\dot{x}, \ddot{x}, \ldots$	the first, second, ... derivatives of x with respect to t
4.11	$\int y \, \mathrm{d}x$	the indefinite integral of y with respect to x
4.12	$\int_a^b y \, \mathrm{d}x$	the definite integral of y with respect to x between the limits $x = a$ and $x = b$

5	Exponential and Logarithmic Functions	
5.1	e	base of natural logarithms
5.2	e^x, $\exp x$	exponential function of x
5.3	$\log_a x$	logarithm to the base a of x
5.4	$\ln x$, $\log_\mathrm{e} x$	natural logarithm of x

6	Trigonometric Functions	
6.1	sin, cos, tan, cosec, sec, cot	the trigonometric functions
6.2	$\left.\begin{array}{l}\sin^{-1}, \cos^{-1}, \tan^{-1} \\ \arcsin, \arccos, \arctan\end{array}\right\}$	the inverse trigonometric functions
6.3	$^\circ$	degrees
6.4	rad	radians

7	Vectors	
7.1	$\mathbf{a}$, $\underline{a}$, $\underset{\sim}{a}$	the vector $\mathbf{a}$, $\underline{a}$, $\underset{\sim}{a}$; these alternatives apply throughout section 9
7.2	$\overrightarrow{\mathrm{AB}}$	the vector represented in magnitude and direction by the directed line segment **AB**
7.3	$\hat{\mathbf{a}}$	a unit vector in the direction of $\mathbf{a}$
7.4	$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the directions of the cartesian coordinate axes
7.5	$\lvert\mathbf{a}\rvert$, a	the magnitude of $\mathbf{a}$
7.6	$\left\lvert\overrightarrow{\mathrm{AB}}\right\rvert$, AB	the magnitude of $\overrightarrow{\mathrm{AB}}$

7	Vectors	
7.7	$\begin{pmatrix} a \\ b \end{pmatrix}$, $a\mathbf{i}+b\mathbf{j}$	column vector and corresponding unit vector notation
7.8	$\mathbf{r}$	position vector
7.9	$\mathbf{s}$	displacement vector
7.10	$\mathbf{v}$	velocity vector
7.11	$\mathbf{a}$	acceleration vector

8	Probability and Statistics	
8.1	A, B, C, etc.	events
8.2	$A \cup B$	union of the events A and B
8.3	$A \cap B$	intersection of the events A and B
8.4	$\mathrm{P}(A)$	probability of the event A
8.5	A'	complement of the event A
8.6	$\mathrm{P}(A \mid B)$	probability of the event A conditional on the event B
8.7	X, Y, R, etc.	random variables
8.8	x, y, r, etc.	values of the random variables X, Y, R etc.
8.9	x_1, x_2, …	observations
8.10	f_1, f_2, …	frequencies with which the observations x_1, x_2, … occur
8.11	$\mathrm{p}(x)$, $\mathrm{P}(X = x)$	probability function of the discrete random variable X
8.12	p_1, p_2, …	probabilities of the values x_1, x_2, … of the discrete random variable X
8.13	$\mathrm{E}(X)$	expectation of the random variable X
8.14	$\mathrm{Var}(X)$	variance of the random variable X
8.15	$\sim$	has the distribution
8.16	$\mathrm{B}(n, p)$	binomial distribution with parameters n and p, where n is the number of trials and p is the probability of success in a trial
8.17	q	$q = 1 - p$ for binomial distribution
8.18	$\mathrm{N}(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2

8	Probability and Statistics	
8.19	$Z \sim \mathrm{N}(0,1)$	standard Normal distribution
8.20	ϕ	probability density function of the standardised Normal variable with distribution $\mathrm{N}(0,\ 1)$
8.21	Φ	corresponding cumulative distribution function
8.22	μ	population mean
8.23	σ^2	population variance
8.24	σ	population standard deviation
8.25	$\overline{x}$	sample mean
8.26	s^2	sample variance
8.27	s	sample standard deviation
8.28	H_0	Null hypothesis
8.29	H_1	Alternative hypothesis
8.30	r	product moment correlation coefficient for a sample
8.31	ρ	product moment correlation coefficient for a population

9	Mechanics	
9.1	kg	kilograms
9.2	m	metres
9.3	km	kilometres
9.4	m/s, $\mathrm{m\,s^{-1}}$	metres per second (velocity)
9.5	$\mathrm{m/s^2}$, $\mathrm{m\,s^{-2}}$	metres per second per second (acceleration)
9.6	F	Force or resultant force
9.7	N	Newton
9.8	N m	Newton metre (moment of a force)
9.9	t	time
9.10	s	displacement
9.11	u	initial velocity
9.12	v	velocity or final velocity
9.13	a	acceleration
9.14	g	acceleration due to gravity
9.15	μ	coefficient of friction

Appendix 3: Use of calculators

Students may use a calculator in all A Level Mathematics examinations. Students are responsible for making sure that their calculators meet the guidelines set out in this appendix.

The use of technology permeates the study of A Level Mathematics. Calculators used **must** include the following features:

- an iterative function
- the ability to compute summary statistics and access probabilities from standard statistical distributions.

In addition, students **must** be told these regulations before sitting an examination:

<table>
<tr>
<td>Calculators must be:
• of a size suitable for use on the desk
• either battery- or solar powered
• free of lids, cases and covers that have printed instructions or formulas.</td>
<td rowspan="2">Calculators must not:
• be designed or adapted to offer any of these facilities
o language translators
o symbolic algebra manipulation
o symbolic differentiation or integration
o communication with other machines or the internet
• be borrowed from another student during an examination for any reason*
• have retrievable information stored in them – this includes
o databanks
o dictionaries
o mathematical formulas
o text.</td>
</tr>
<tr>
<td>The student is responsible for the following:
• the calculator’s power supply
• the calculator’s working condition
• clearing anything stored in the calculator.</td>
</tr>
</table>

Advice: *an invigilator may give a student a replacement calculator.

Appendix 4: Assessment Objectives

The following tables outline in detail the strands and elements of each Assessment Objective for A Level Mathematics, as provided by Ofqual in the document *GCE Subject Level Guidance for Mathematics.*

- A 'strand' is a discrete bullet point that is formally part of an assessment objective
- An 'element' is an ability that the assessment objective does not formally separate, but that could be discretely targeted or credited.

AO1: Use and apply standard techniques. **Learners should be able to:** ▪ **select and correctly carry out routine procedures** ▪ **accurately recall facts, terminology and definitions**	***50% (A Level)*** ***60% (AS)***
Strands	**Elements**
1. select and correctly carry out routine procedures	1a – select routine procedures
	1b – correctly carry out routine procedures
2. accurately recall facts, terminology and definitions	This strand is a single element

AO2: Reason, interpret and communicate mathematically **Learners should be able to:** ▪ **construct rigorous mathematical arguments (including proofs)** ▪ **make deductions and inferences** ▪ **assess the validity of mathematical arguments** ▪ **explain their reasoning** ▪ **use mathematical language and notation correctly**	***25% (A Level)*** ***20% (AS)***
Strands	**Elements**
1. construct rigorous mathematical arguments (including proofs)	This strand is a single element
2. make deductions and inferences	2a – make deductions
1.	2b – make inferences
3. assess the validity of mathematical arguments	This strand is a single element
4. explain their reasoning	This strand is a single element
5. use mathematical language and notation correctly	This strand is a single element

AO3: Solve problems within mathematics and in other contexts **Learners should be able to:** ▪ **translate problems in mathematical and non-mathematical contexts into mathematical processes** ▪ **interpret solutions to problems in their original context, and, where appropriate, evaluate their accuracy and limitations** ▪ **translate situations in context into mathematical models** ▪ **use mathematical models** ▪ **evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them**	***25% (A Level)*** ***20% (AS)***
Strands	**Elements**
1. translate problems in mathematical and non-mathematical contexts into mathematical processes	1a – translate problems in mathematical contexts into mathematical processes
1.	1b – translate problems in non-mathematical contexts into mathematical processes
2. interpret solutions to problems in their original context, and, where appropriate evaluate their accuracy and limitations	2a – interpret solutions to problems in their original context
1.	2b – where appropriate, evaluation the accuracy and limitations of solutions to problems
3. translate situations in context into mathematical models	This strand is a single element
4. use mathematical models	This strand is a single element
5. evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them	5a – evaluate the outcomes of modelling in context
	5b – recognise the limitations of models
	5c – where appropriate, explain how to refine models

Assessment Objectives coverage

There will be full coverage of all elements of the Assessment Objectives, with the exception of AO3.2b and AO3.5c, in each set of A Level Mathematics assessments offered by Pearson. Elements AO3.2b and AO3.5c will be covered in each route through the qualification within three years.

Appendix 5: The context for the development of this qualification

All our qualifications are designed to meet our World Class Qualification Principles[1] and our ambition to put the student at the heart of everything we do.

We have developed and designed this qualification by:

- reviewing other curricula and qualifications to ensure that it is comparable with those taken in high-performing jurisdictions overseas
- consulting with key stakeholders on content and assessment, including learned bodies, subject associations, higher-education academics, teachers and employers to ensure this qualification is suitable for a UK context
- reviewing the legacy qualification and building on its positive attributes.

This qualification has also been developed to meet criteria stipulated by Ofqual in their documents *GCE Qualification Level Conditions and Requirements* and *GCE Subject Level Conditions and Requirements for Mathematics*, published in April 2016.

[1] Pearson's World Class Qualification Principles ensure that our qualifications are:

- **demanding**, through internationally benchmarked standards, encouraging deep learning and measuring higher-order skills
- **rigorous**, through setting and maintaining standards over time, developing reliable and valid assessment tasks and processes, and generating confidence in end users of the knowledge, skills and competencies of certified students
- **inclusive**, through conceptualising learning as continuous, recognising that students develop at different rates and have different learning needs, and focusing on progression
- **empowering**, through promoting the development of transferable skills, see *Appendix 6.*

From Pearson's Expert Panel for World Class Qualifications

May 2014

"The reform of the qualifications system in England is a profoundly important change to the education system. Teachers need to know that the new qualifications will assist them in helping their learners make progress in their lives.

When these changes were first proposed we were approached by Pearson to join an 'Expert Panel' that would advise them on the development of the new qualifications.

We were chosen, either because of our expertise in the UK education system, or because of our experience in reforming qualifications in other systems around the world as diverse as Singapore, Hong Kong, Australia and a number of countries across Europe.

We have guided Pearson through what we judge to be a rigorous qualification development process that has included:

- extensive international comparability of subject content against the highest-performing jurisdictions in the world
- benchmarking assessments against UK and overseas providers to ensure that they are at the right level of demand
- establishing External Subject Advisory Groups, drawing on independent subject-specific expertise to challenge and validate our qualifications
- subjecting the final qualifications to scrutiny against the DfE content and Ofqual accreditation criteria in advance of submission.

Importantly, we have worked to ensure that the content and learning is future oriented. The design has been guided by what is called an 'Efficacy Framework', meaning learner outcomes have been at the heart of this development throughout.

We understand that ultimately it is excellent teaching that is the key factor to a learner's success in education. As a result of our work as a panel we are confident that we have supported the development of qualifications that are outstanding for their coherence, thoroughness and attention to detail and can be regarded as representing world-class best practice."

Sir Michael Barber (Chair)

Chief Education Advisor, Pearson plc

Professor Lee Sing Kong

Director, National Institute of Education, Singapore

Bahram Bekhradnia

President, Higher Education Policy Institute

Professor Jonathan Osborne

Stanford University

Dame Sally Coates

Principal, Burlington Danes Academy

Professor Dr Ursula Renold

Federal Institute of Technology, Switzerland

Professor Robin Coningham

Pro-Vice Chancellor, University of Durham

Professor Bob Schwartz

Harvard Graduate School of Education

Dr Peter Hill

Former Chief Executive ACARA

All titles correct as at May 2014

Appendix 6: Transferable skills

The need for transferable skills

In recent years, higher education institutions and employers have consistently flagged the need for students to develop a range of transferable skills to enable them to respond with confidence to the demands of undergraduate study and the world of work.

The Organisation for Economic Co-operation and Development (OECD) defines skills, or competencies, as 'the bundle of knowledge, attributes and capacities that can be learned and that enable individuals to successfully and consistently perform an activity or task and can be built upon and extended through learning.' [1]

To support the design of our qualifications, the Pearson Research Team selected and evaluated seven global 21st-century skills frameworks. Following on from this process, we identified the National Research Council's (NRC) framework as the most evidence-based and robust skills framework. We adapted the framework slightly to include the Program for International Student Assessment (PISA) ICT Literacy and Collaborative Problem Solving (CPS) Skills.

The adapted National Research Council's framework of skills involves: [2]

Cognitive skills

- **Non-routine problem solving** – expert thinking, metacognition, creativity.
- **Systems thinking** – decision making and reasoning.
- **Critical thinking** – definitions of critical thinking are broad and usually involve general cognitive skills such as analysing, synthesising and reasoning skills.
- **ICT literacy** – access, manage, integrate, evaluate, construct and communicate. [3]

Interpersonal skills

- **Communication** – active listening, oral communication, written communication, assertive communication and non-verbal communication.
- **Relationship-building skills** – teamwork, trust, intercultural sensitivity, service orientation, self-presentation, social influence, conflict resolution and negotiation.
- **Collaborative problem solving** – establishing and maintaining shared understanding, taking appropriate action, establishing and maintaining team organisation.

Intrapersonal skills

- **Adaptability** – ability and willingness to cope with the uncertain, handling work stress, adapting to different personalities, communication styles and cultures, and physical adaptability to various indoor and outdoor work environments.
- **Self-management and self-development** – ability to work remotely in virtual teams, work autonomously, be self-motivating and self-monitoring, willing and able to acquire new information and skills related to work.

Transferable skills enable young people to face the demands of further and higher education, as well as the demands of the workplace, and are important in the teaching and learning of this qualification. We will provide teaching and learning materials, developed with stakeholders, to support our qualifications.

---

[1] OECD – *Better Skills, Better Jobs, Better Lives* (OECD Publishing, 2012)

[2] Koenig J A, National Research Council – *Assessing 21st Century Skills: Summary of a Workshop* (National Academies Press, 2011)

[3] PISA – *The PISA Framework for Assessment of ICT Literacy* (2011)

Appendix 7: Level 3 Extended Project qualification

What is the Extended Project?

The Extended Project is a standalone qualification that can be taken alongside GCEs. It supports the development of independent learning skills and helps to prepare students for their next step – whether that be higher education or employment. The qualification:

- is recognised by higher education for the skills it develops
- is worth half of an Advanced GCE qualification at grades A*–E
- carries UCAS points for university entry.

The Extended Project encourages students to develop skills in the following areas: research, critical thinking, extended writing and project management. Students identify and agree a topic area of their choice for in-depth study (which may or may not be related to a GCE subject they are already studying), guided by their teacher.

Students can choose from one of four approaches to produce:

- a dissertation (for example an investigation based on predominately secondary research)
- an investigation/field study (for example a practical experiment)
- a performance (for example in music, drama or sport)
- an artefact (for example creating a sculpture in response to a client brief or solving an engineering problem).

The qualification is coursework based and students are assessed on the skills of managing, planning and evaluating their project. Students will research their topic, develop skills to review and evaluate the information, and then present the final outcome of their project.

The Extended Project has 120 guided learning hours (GLH) consisting of a 40-GLH taught element that includes teaching the technical skills (for example research skills) and an 80-GLH guided element that includes mentoring students through the project work. The qualification is 100% internally assessed and externally moderated.

How to link the Extended Project with mathematics

The Extended Project creates the opportunity to develop transferable skills for progression to higher education and to the workplace, through the exploration of either an area of personal interest or a topic of interest from within the mathematics qualification content.

Through the Extended Project, students can develop skills that support their study of mathematics, including:

- conducting, organising and using research
- independent reading in the subject area
- planning, project management and time management
- defining a hypothesis to be tested in investigations or developing a design brief
- collecting, handling and interpreting data and evidence
- evaluating arguments and processes, including arguments in favour of alternative interpretations of data and evaluation of experimental methodology
- critical thinking.

In the context of the Extended Project, critical thinking refers to the ability to identify and develop arguments for a point of view or hypothesis, and to consider and respond to alternative arguments.

Types of Extended Project related to mathematics

Students may produce a dissertation on any topic that can be researched and argued. In mathematics this might involve working on a substantial statistical project or a project that requires the use of mathematical modelling.

Projects can give students the opportunity to develop mathematical skills that cannot be adequately assessed in examination questions.

- ***Statistics*** – students can have the opportunity to plan a statistical enquiry project, use different methods of sampling and data collection, use statistical software packages to process and investigate large quantities of data and review results to decide if more data is needed.
- ***Mathematical modelling*** – students can have the opportunity to choose modelling assumptions, compare with experimental data to assess the appropriateness of their assumptions and refine their modelling assumptions until they get the required accuracy of results.

Using the Extended Project to support breadth and depth

In the Extended Project, students are assessed on the quality of the work they produce and the skills they develop and demonstrate through completing this work. Students should demonstrate that they have extended themselves in some significant way beyond what they have been studying in mathematics. Students can demonstrate extension in one or more dimensions:

- ***deepening understanding*** – where a student explores a topic in greater depth than in the specification content. This could be an in-depth exploration of one of the topics in the specification
- ***broadening skills*** – where a student learns a new skill. This might involve learning the skills in statistics or mathematical modelling mentioned above or learning a new mathematical process and its practical uses
- ***widening perspectives*** – where the student's project spans different subjects. Projects in a variety of subjects need to be supported by data and statistical analysis. Students studying mathematics with design and technology can carry out design projects involving the need to model a situation mathematically in planning their design.

A wide range of information to support the delivery and assessment of the Extended Project, including the specification, teacher guidance for all aspects, an editable scheme of work and exemplars for all four approaches, can be found on our website.

Appendix 8: Codes

Type of code	Use of code	Code
Discount codes	Every qualification eligible for performance tables is assigned a discount code indicating the subject area to which it belongs. Discount codes are published by the DfE.	Please see the GOV.UK website*
Regulated Qualifications Framework (RQF) codes	Each qualification title is allocated an Ofqual Regulated Qualifications Framework (RQF) code. The RQF code is known as a Qualification Number (QN). This is the code that features in the DfE Section 96 and on the LARA as being eligible for 16–18 and 19+ funding, and is to be used for all qualification funding purposes. The QN will appear on students' final certification documentation.	The QN for this qualification is: 603/1333/X
Subject codes	The subject code is used by centres to enter students for a qualification. Centres will need to use the entry codes only when claiming students' qualifications.	A Level – 9MA0
Paper codes	These codes are provided for reference purposes. Students do not need to be entered for individual papers.	Paper 1: 9MA0/01 Paper 2: 9MA0/02 Paper 3: 9MA0/03

*https://www.gov.uk/government/publications/key-stage-4-qualifications-discount-codes-and-point-scores

TM22/01/20 9781446957097_GCE2017_AL_MATHS_ISSUE 4.DOC.1-74/0

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Origami photography: Pearson Education Ltd/Naki Kouyioumtzis

ISBN 978 1 446 95709 7

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ISBN 978-1-4469-5709-7
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VIDEO LINKS

A2 Maths Tutorials - Pure

A2 Maths Tutorials - Applied

A2 Maths - Revision Resources

Revision Self Evaluation for A level maths

January Exam - Revision Materials

Revision Sheet 1 Teacher 1
Revision Sheet 1 Teacher 2
Revision Sheet 2 Teacher 1
Revision Sheet 2 Teacher 2
Revision Sheet 3 Teacher 1
Revision Sheet 3 Teacher 2
Revision Sheet 4 Teacher 1
Revision Sheet 4 Teacher 2
Revision Sheet 5 Teacher 1
Revision Sheet 5 Teacher 2
Revision Sheet 6 Teacher 1
Revision Sheet 6 Teacher 2

Summer Exam Revision Materials - Pure

Practice Paper A [SS]
Practice Paper B [SS]
Practice Paper C [SS]
Practice Paper D [SS]
Practice Paper E [SS]
Practice Paper F [SS]

Specimen Paper 1
Specimen Paper 2

Mock Paper - Set 2 - Paper 1 [MS]
Mock Paper - Set 2 - Paper 2 [MS]

June 2018 - Paper 1
June 2018 - Paper 2
June 2019 - Paper 1
June 2019 - Paper 2

Pure Practice Topic Tests by Edexcel

Proof [MS]
Algebraic and Partial Fractions [MS]
Functions and Modelling [MS]
Sequences and Series [MS]
The Binomial Theorem [MS]
Trigonometry [MS]
Parametric Equations [MS]
Differentiation [MS]
Numerical Methods [MS]
Integration 1 [MS]
Integration 2 [MS]
Vectors [MS]

Summer Exam Revision Materials - Applied

Practice Paper G [SS1] [SS2]
Practice Paper H [SS1] [SS2]
Practice Paper I [SS1] [SS2]

Specimen Paper 3

Mock Paper - Set 2 - Paper 3 - Stats [MS]
Mock paper - Set 2 - Paper 3 - Mech [MS]

June 2018 - Paper 3
June 2019 - Paper 3

Applied Practice Topic Tests by Edexcel

Regression and Correlation [MS]
Probability [MS]
Normal Distribution [MS]
Moments [MS]
Forces at an angle [MS]
Projectiles [MS]
Applications of forces [MS]
Calculus and Vectors in Mechanics [MS]

Old Spec Resources - Pure - Past Papers (with video solutions)

Some topics are missing and extra topics have been ruled out. 90% good

Pure Paper 2012
Pure Paper 2013
Pure Paper 2014
Pure Paper 2015
Pure Paper 2016
Pure Paper 2017
Pure Paper 2018

Old Spec - Pure - Bronze/Silver/Gold Papers

Some topics are missing and extra topics have been ruled out. 90% good

Bronze 1A	Bronze 1B
Bronze 2A	Bronze 2B
Bronze 3A	Bronze 3B
Bronze 4A	Bronze 4B
Bronze 5A	Bronze 5B
Silver 1A	Silver 1B
Silver 2A	Silver 2B
Silver 3A	Silver 3B
Silver 4A	Silver 4B
Silver 5A	Silver 5B
Gold 1A	Gold 1B
Gold 2A	Gold 2B
Gold 3A	Gold 3B
Gold 4A	Gold 4B
Gold 5A	

SS = Staff Solutions, SS1 = Stats, SS2 = Mech

Useful Links

Parametric Equations

- https://mmerevise.co.uk/a-level-maths-revision/parametric-equations/
- **https://www.savemyexams.com/a-level/maths_pure/aqa/18/revision-notes/9-parametric-equations/9-1-parametric-equations/9-1-1-parametric-equations---basics/**
- **https://alevelmaths.co.uk/pure-maths/algebra/parametric-equations/**
- **https://senecalearning.com/en-GB/revision-notes/a-level/maths/edexcel/pure-maths/3-3-2-modelling-with-parametric-equations**
- **https://www2.clarku.edu/faculty/djoyce/trig/identities.html**
- **https://www.examsolutions.net/tutorials/exam-questions-arithmetic-sequences-and-series/**

REVISION NOTES

A LEVEL PURE MATHS REVISON NOTES

1 ALGEBRA AND FUNCTIONS

a) <u>INDICES</u>

Rules to learn :

$x^a \times x^b = x^{a+b}$ $\quad x^a \div x^b = x^{a-b}$ $\quad (x^a)^b = x^{ab}$ $\quad x^{-a} = \frac{1}{x^a}$ $\quad x^{\frac{n}{m}} = \sqrt[m]{x^n} = \left(\sqrt[m]{x}\right)^n$

Simplify $2x(x-y)^{\frac{3}{2}} + 3(x-y)^{\frac{1}{2}}$

$= (x-y)^{\frac{1}{2}}(2x(x-y)+3))$

$= (x-y)^{\frac{1}{2}}(2x^2 - 2xy + 3)$

Solve $3^{2x} \times 25^x = 15$

$(3 \times 5)^{2x} = 15^1$

$2x = 1$

$x = \frac{1}{2}$

b) <u>SURDS</u>

- A root such as $\sqrt{3}$ that cannot be written as a fraction is IRRATIONAL
- An expression that involves irrational roots is in SURD FORM
- RATIONALISING THE DENOMINATOR is removing the surd from the denominator (multiply by the conjugate)

Simplify

$\sqrt{75} - \sqrt{12}$

$= \sqrt{5 \times 5 \times 3} - \sqrt{2 \times 2 \times 3}$

$= 5\sqrt{3} - 2\sqrt{3}$

$= 3\sqrt{3}$

Rationalise the denominator $\frac{2}{2-\sqrt{3}}$

$= \frac{2}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$

$= 4 + 2\sqrt{3}$

The conjugate of the denominator $2 - \sqrt{3}$ is $2 + \sqrt{3}$ so that $(2 - \sqrt{3})(2 + \sqrt{3})$

$= 2^2 - \sqrt{3}^2$

$= 1$

c) <u>QUADRATIC EQUATIONS AND GRAPHS</u>

Factorising – identifying the roots of the equation $ax^2 + bx + c = 0$

- Look for the **difference of 2 squares** $x^2 - a^2 = (x + a)(x - a)$ or $(ax)^2 - b^2 = (ax + b)(ax - b)$
- Look for the perfect square $x^2 + 2ax + a^2 = (x + a)^2$
- Look out for equations which can be transformed into quadratic equations

Solve $x + 1 - \frac{12}{x} = 0$

$x^2 + x - 12 = 0$

$(x+4)(x-3) = 0$

x = -4 x = 3

Solve $6x^4 - 7x^2 + 2 = 0$

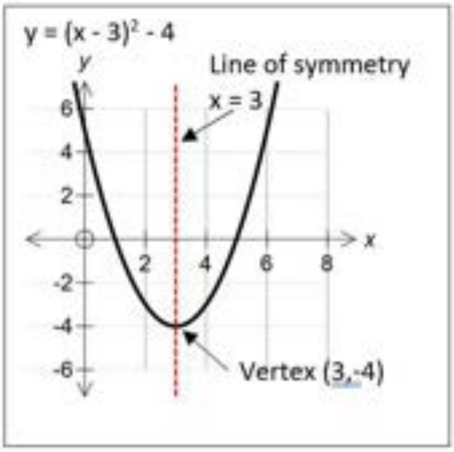
Let z = x^2 $\quad 6z^2 - 7z + 2 = 0$

$(2z-1)(3z-2) = 0$

$z = \frac{1}{2} \quad x = \pm\sqrt{\frac{1}{2}} \qquad z = \frac{2}{3} \quad x = \pm\sqrt{\frac{2}{3}}$

Completing the square – identifying the vertex and line of symmetry

$y = (x + a)^2 + b$ vertex at (-a , b) line of symmetry as equation x = -a



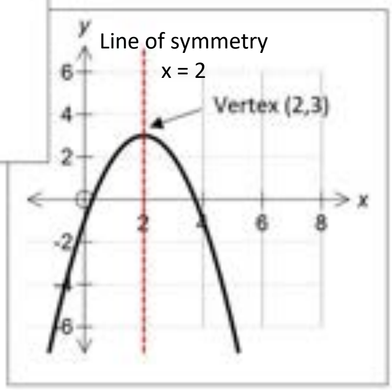
Sketch the graph of

$y = 4x - x^2 - 1$

$y = -(x^2 - 4x) - 1$

$y = -((x-2)^2 - 4) - 1$

$y = -(x-2)^2 + 3$



Quadratic formula (and the DISCRIMINANT)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ for solving $ax^2 + bx + c = 0$

The **DISCRIMINANT $b^2 - 4ac$** can be used to identify the number of roots

$b^2 - 4ac > 0$ there are 2 real distinct roots (graph crosses the x-axis twice)

$b^2 - 4ac = 0$ there is a single repeated root (the x-axis is a tangent)

$b^2 - 4ac < 0$ there are no real roots (the graph does not touch the x-axis)

d) SIMULTANEOUS EQUATIONS

Solving by elimination

$3x - 2y = 19 \quad \times 3 \quad 9x - \mathbf{6y} = 57$

$2x - 3y = 21 \quad \times 2 \quad \underline{4x - \mathbf{6y} = 42}$

$5x - 0y = 15 \qquad x = 3 \quad (9 - 2y = 19) \qquad y = -5$

Solving by substitution

$x + y = 1 \quad (y = 1 - x)$

$x^2 + y^2 = 25 \qquad x^2 + (1 - x)^2 = 25$

$2x^2 - 2x - 24 = 0$

$2(x - 4)(x + 3) = 0 \qquad x = 4 \quad y = -3 \qquad x = -3 \quad y = 4$

If you end up with a quadratic equation when solving simultaneously the discriminant can be used to determine the relationship between the graphs

If $b^2 - 4ac > 0$ the graphs intersect at 2 distinct points

$b^2 - 4ac = 0$ the graphs intersect at 1 point (or tangent)

$b^2 - 4ac < 0$ the graphs do not intersect

e) INQUALITIES

Linear Inequality - solve using the same method as solving a linear equation but remember to reverse the inequality if you multiply or divide by a negative number

Quadratic Inequality – always a good idea to sketch a graph

≤ ≥ plot the graph as a solid line or curve

< > plot as a dotted/dashed line or curve

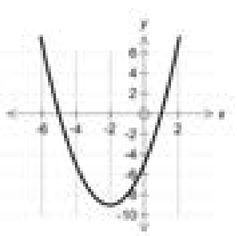
If you are unsure of which area to shade pick a point in one of the regions and check the inequalities using the coordinates of the point

Solve **$x^2 + 4x - 5 < 0$**

$x^2 + 4x - 5 = 0$

$(x - 1)(x + 5) = 0$

$x = 1 \quad x = -5$



$x^2 + 4x - 5 < 0$

$-5 < x < 1$

which can be written as

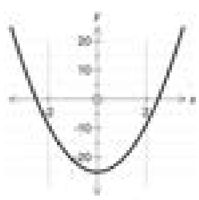
$\{x : x > -5\} \cap \{x : x < 1\}$

Solve **$4x^2 - 25 \geq 0$**

$4x^2 - 25 = 0$

$(2x - 5)(2x + 5) = 0$

$x = \frac{5}{2} \quad x = -\frac{5}{2}$



$4x^2 - 25 \geq 0$

$x \leq \frac{5}{2}$ or $x \geq \frac{5}{2}$

which can be written as

$\{x : x \leq -\frac{5}{2}\} \cup \{x : x \geq \frac{5}{2}\}$

f) POLYNOMIALS

- A polynomial is an expression which can be written in the form $ax^n + bx^{n-1} + cx^{n-2} + \ldots$ where a,b, c are constants and n is a positive integer.
- The **order** of the polynomial is the highest power of x in the polynomial
- Polynomials can be divided to give a **Quotient** and **Remainder**

Divide $x^3 - x^2 + x + 15$ by $x + 2$

$$
\begin{array}{r|rrrr}
 & x^2 & -3x & +7 & \leftarrow \text{Quotient} \\
\hline
x+2 & x^3 - x^2 & +x & +15 & \\
 & x^3 + 2x^2 & & & \\
\hline
 & -3x^2 & +x & & \\
 & -3x^2 & -6x & & \\
\hline
 & & 7x & +15 & \\
 & & 7x & +14 & \\
\hline
 & & & 1 & \leftarrow \text{Remainder}
\end{array}
$$

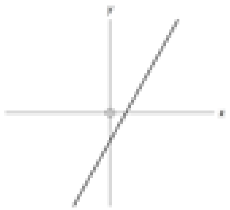
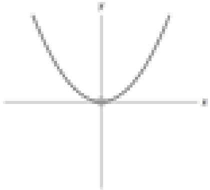
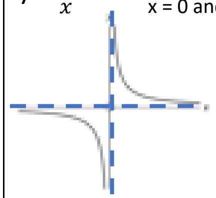
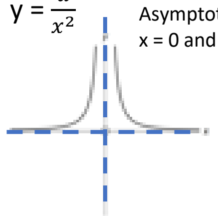
- **Factor Theorem** – If (x – a) is a factor of f(x) then f(a) = 0 and is root of the equation f(x) = 0

Show that (x – 3) is a factor of $x^3 - 19x + 30 = 0$
$f(x) = x^3 - 19x + 30$
$f(3) = 3^3 - 19 \times 3 + 20$
$= 0$
f(3) = 0 so x – 3 is a factor of f(x)

g) GRAPHS OF FUNCTIONS

Sketching Graphs

- Identify where the graph crossed the y-axis (x = 0)
- Identify where the graph crossed the x-axis (y = 0)
- Identify any asymptotes and plot with a dashed line

$y=mx + c$	$y = kx^2$	$y=kx^3$	$y = \frac{a}{x}$ Asymptotes at x = 0 and y = 0	$y = \frac{a}{x^2}$ Asymptotes at x = 0 and y = 0
				
	y is proportional to x^2	y is proportional to x^3	y is proportional to $\frac{k}{x}$	y is proportional to $\frac{a}{x^2}$

Modulus Graphs

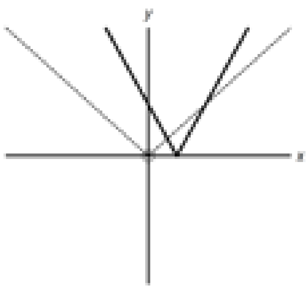
- |x| is the 'modulus of x' or the absolute value |2|=2 |-2|= 2
- To sketch the graph of y = |f(x)| sketch y = f(x) and take any part of the graph which is below the x-axis and reflect it in the x-axis

Solve $|2x - 4| < |x|$

$2x - 4 = x$
$x = 4$

$2x - 4 = -x$
$3x = 4$
$x = \frac{4}{3}$

$\frac{4}{3} < x < 4$



h) FUNCTIONS

- A function is a rule which generates exactly ONE OUTPUT for EVERY INPUT
- **DOMAIN** – defines the set of the values that can be 'put into' the function   f(x) = $\sqrt{x}$ domain x ≥ 0
- **RANGE** – defines the set of values 'output' by the function – make sure it is defined in terms of f(x) and not x

  $f : x \mapsto x^2 \quad x \in \mathbb{R}$ means an input a is converted to a^2 where the input 'a' can be any real number

  $Range\ f(x) \geq 0$

- **INVERSE FUNCTION** denoted by $f^{-1}(x)$

  The domain of $f^{-1}(x)$ is the range of f(x)

  The range of $f^{-1}(x)$ is the domain of f(x)

  Using the same scale on the x and y axis the graphs of a function and it's inverse have **reflection symmetry** in the line **y = x**

$f(x) = \frac{3}{x+2}$ find $f^{-1}(x)$

$y = \frac{3}{x+2}$

$x = \frac{3}{y} - 2$

$f^{-1}(x) = \frac{3}{x} - 2$

- **COMPOSITE FUNCTIONS**

  The function **gf**(x) is a composite function which tells you 'to do' **f** first and then use the output in **g**

f(x) = 4x   g(x) = $x^2 - 1$

fg(x) = $4(x^2 -1)$
= $4x^2 - 4$

gf(x) = $(4x)^2 - 1$
= $16x^2 - 1$

i) TRANSFORMING GRAPHS

Translation

To find the equation of a graph after a translation of $\begin{bmatrix} a \\ b \end{bmatrix}$ replace x by (x – a) and y by (y – b)

y = f(x -a) + b

The graph of y = x^2 -1 is translated by $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

Find the equation of the resulting graph.

$(y + 2) = (x – 3)^2 - 1$

$y = x^2 – 6x + 6$

Reflection

Reflection in the x-axis replace y with -y   y = -f(x)

Reflection in the y-axis replace x with -x   y = f(-x)

Stretch

Stretch in the y-direction by scale factor a   y = af(x)

Stretch on the x-direction by scale factor $\frac{1}{a}$ y = f(ax)

Combining Transformations

Take care with the order in which the transformations are carried out.

The graph of y = x^2 **is translated by** $\begin{bmatrix} \mathbf{3} \\ \mathbf{0} \end{bmatrix}$ and then **reflected in the y axis**. Find the equation of the resulting graph

Translation $y = (x – 3)^2$
$= x^2 -6x + 9$

Reflection $y = (-x)^2 -6(-x) + 9$
$\mathbf{= x^2 + 6x + 9}$

The graph of y = x^2 is **reflected in the y axis** and then **translated by** $\begin{bmatrix} \mathbf{3} \\ \mathbf{0} \end{bmatrix}$. Find the equation of the resulting graph

Reflection $y = (-x)^2$
$= x^2$

Translation $y = (x – 3)^2$
$\mathbf{= x^2 - 6x + 9}$

j) PARTIAL FRACTIONS

Any proper algebraic fractions with a denominator that is a product of linear factors can be written as partial fractions

- Useful for integrating a rational function
- Useful for finding binomial approximations

$$\frac{px+q}{(ax+b)(cx+d)(ex+f)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f} \qquad \frac{px+q}{(ax+b)(cx+d)^2} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$$

Express $\frac{5}{(x-2)(x+3)}$ in the form $\frac{A}{x-2} + \frac{B}{x+3}$

$\frac{A}{x-2} + \frac{B}{x+3} \equiv \frac{A(x+3)+B(x-2)}{(x+3)(x-2)}$

A(x + 3) + B(x - 2) = 5   x = 2   5A = 5   x = -3   -5B = 5

A = 1   B = -1

$$\frac{5}{(x-2)(x+3)} = \frac{1}{x-2} - \frac{1}{x+3}$$

2 COORDINATE GEOMETRY

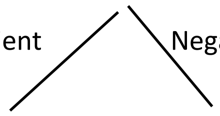
a) Graphs of linear functions

y = mx + c

the line intercepts the y axis at (0, c)

Gradient = $\frac{change\ in\ y}{change\ in\ x}$

Positive gradient



Negative gradient

Finding the equation of a line with gradient m through point (x_1, y_1)

Use the equation $(y - y_1) = m(x - x_1)$

If necessary rearrange to the required form (ax + by = c or y = mx + c)

Parallel and Perpendicular Lines

$y = m_1x + c_1 \qquad y = m_2x + c_2$

If $\mathbf{m_1 = m_2}$ then the lines are **PARALLEL**

If $\mathbf{m_1 \times m_2 = -1}$ then the lines are **PERPENDICULAR**

Find the equation of the line perpendicular to the line y – 2x = 7 passing through point (4, -6)

Gradient of y – 2x = 7 is 2 (y = 2x + 7)

Gradient of the perpendicular line = - ½   (2 × -½ = -1)

Equation of the line with gradient –½ passing through (4, -6)

(y + 6) = -½(x – 4)

2y + 12 = 4 – x

x + 2y = - 8

Finding the mid-point of the line segment joining (a,b) and (c,d)

Mid-point = $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$

Calculating the length of a line segment joining (a,b) and (c,d)

Length = $\sqrt{(c-a)^2 + (d-b)^2}$

b) Circles

A circle with centre (0,0) and radius r has the equations $x^2 + y^2 = r^2$

A circle with centre (a,b) and radius r is given by $(x - a)^2 + (y - b)^2 = r^2$

Finding the centre and the radius (completing the square for x and y)

Find the centre and radius of the circle $x^2 + y^2 + 2x - 4y - 4 = 0$

$x^2 + 2x + y^2 - 4y - 4 = 0$

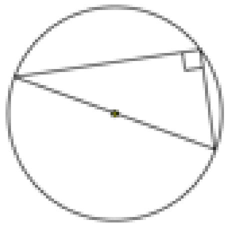
$(x + 1)^2 - 1 + (y - 2)^2 - 4 - 4 = 0$

$(x + 1)^2 + (y - 2)^2 = 3^2$

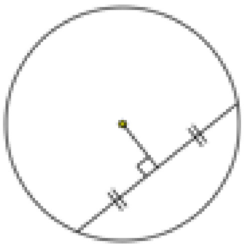
Centre (-1, 2)   Radius = 3

The following circle properties might be useful

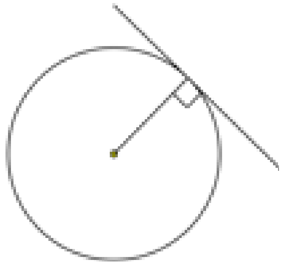
Angle in a semi-circle is a right angle



The perpendicular from the centre to a chord bisects the chord



The tangent to a circle is perpendicular to the radius



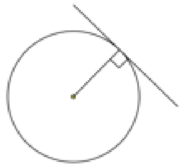
Finding the equation of a tangent to a circle at point (a,b)

The gradient of the tangent at (a,b) is perpendicular to the gradient of the radius which meets the circumference at (a, b)

Find equation of the tangent to the circle $x^2 + y^2 - 2x - 2y - 23 = 0$ at the point (5,4)

$(x - 1)^2 + (y - 1)^2 - 25 = 0$

Centre of the circle (1,1)



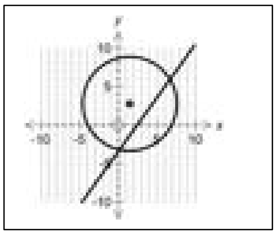
Gradient of radius = $\frac{4-1}{5-1} = \frac{3}{4}$ Gradient of tangent = $-\frac{4}{3}$

Equation of the tangent $(y - 4) = -\frac{4}{3}(x - 5)$ $3y - 12 = 20 - 4x$

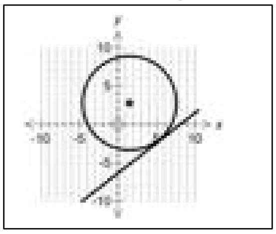
$4x + 3y = 32$

Lines and circles Solving simultaneously to investigate the relationship between a line and a circle will result in a quadratic equation. **Use the discriminant to determine the relationship between the line and the circle**

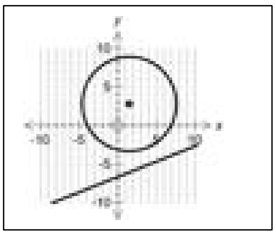
$b^2 - 4ac > 0$



$b^2 - 4ac = 0$ (tangent)



$b^2 - 4ac < 0$



c) Parametric Equations

- Two equations that separately define the x and y coordinates of a graph in terms of a third variable
- The third variable is called the **parameter**
- To convert a pair of **parametric equations** to a **cartesian equation** you need to eliminate the parameter (you may need to use trig identities if the parametric equations involve trig functions)

Find the cartesian equation of the curve given by the parametric equations given by $x = cos\theta \quad y = sin2\theta$

$y = sin2\theta$

$y = 2sin\theta cos\theta \qquad y^2 = 4sin^2\theta cos^2\theta$

$= 4(1 - cos^2\theta)cos^2\theta$

$y^2 = 4(1 - x^2)x^2$

3. SEQUENCES AND SERIES

a) Binomial

Expansion of $(1 + x)^n \quad |x| < 1 \quad n \in \mathbb{Q}$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2}x^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}x^3 \ldots\ldots\ldots\ldots + nx^{n-1} + x^n$$

Use the binomial expansion to write down the first four terms of $\frac{1}{(2-3x)^2}$

$$2^{-2}\left(1 - \frac{3}{2}x\right)^{-2} = 2^{-2}(1 + -2 \times (-\frac{3}{2}x) + \frac{-2\times-3}{1\times2}\left(-\frac{3}{2}x\right)^2 + \frac{-2\times-3\times-4}{1\times2\times3}(-\frac{3}{2}x)^3$$

$$= \frac{1}{4}(1 + 3x + \frac{27}{4}x^2 + \frac{27}{2}x^3)$$

$$= \frac{1}{4} + \frac{3}{4}x + \frac{27}{16}x^2 + \frac{27}{8}x^3$$

Expansion of $(a + b)^n \quad n \in \mathbb{Z}^+$

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\times2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\times2\times3}a^{n-3}b^3 \ldots\ldots\ldots\ldots + nab^{n-1} + b^n$$

Find the coefficient of the x^3 term in the expansion of $(2 + 3x)^9$

$(3x)^3$ must have 2^6 as part of the coefficient ($^{3 + 6 = 9}$)

$\frac{9\times8\times7}{1\times2\times3} \times 2^6 \times (3x)^3$ = 145152 (x^3)

b) Sequences

- An inductive definition defines a sequence by giving the first term and a rule to find the next term(s)
  $u_{n+1} = f(u_n) \qquad u_1 = a$

Find the first 3 terms of a sequence defined by $u_{n+1} = 2u_n + 1 \qquad u_1 = 2$

$u_1 = 2 \quad u_2 = 2 \times 2 + 1 \qquad u_3 = 2 \times 5 + 1$

= 5 = 11

- An **increasing** sequence is one where $u_{n+1} > u_n$ for all n
- An **decreasing** sequence is one where $u_{n+1} < u_n$ for all n
- A sequence may converge to a **limit L** $u_{n+1} = f(u_n)$ as $n \to \infty \qquad u_{n+1} = u_n = L$

The sequence defined by $u_{n+1} = 0.2u_n + 2 \quad u_1 = 3$ converges to a limit L . Find L

L = 0.2L + 2 0.8L = 2 L = 2.5

- A periodic sequence repeats itself over a fixed interval
  $u_{n+a} = u_n$ for all n for a constant a which is the period of the sequence

c) Sigma Notation – sum of

$\sum_{r=1}^{6}(r^2 + 1)$ = (1^2+1) + (2^2+1) + (3^2+1) + (4^2+1) + (5^2+1) + (6^2+1)

= 2 + 5 + 10 + 17 + 26 + 37

= 97

Staring with the 1st term r = 1
Ending with the 6th term r = 6

d) Arithmetic sequences and series

- Each term is found by adding a fixed constant (**common difference d)** to the previous term
- The first term is **a** giving the sequence a , a + d, a + 2d, a + 3d…… where $u_n = a + (n-1)d$
- The sum of the first n terms can be found using:

$$S_n = \frac{n}{2}(2a + (n-1)d) \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$ where *l* is the last term

e) Geometric sequence and series

- Each term is found by multiplying the previous term by a fixed constant (**common ratio r)**
- The first term is **a** giving the sequence a, ar, ar^2, ar^3, ar^4…
- The sum of the first n terms can be found using

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{or} \quad S_n = \frac{a(r^n-1)}{r-1} \qquad S_\infty = \frac{a}{1-r} \quad |r| < 1$$

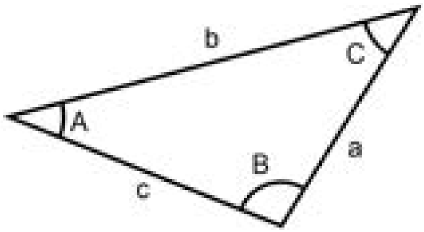
4. TRIGONOMETRY

MAKE SURE YOU KNOW AND CAN USE THE FOLLOWING FROM GCSE

$$Area = \frac{1}{2}absinC$$

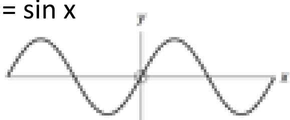
$$\frac{a}{SinA} = \frac{b}{SinB} = \frac{c}{SinC} \quad \text{or} \quad \frac{SinA}{a} = \frac{SinB}{b} = \frac{SinC}{c}$$

$$a^2 = b^2 + c^2 - 2bcCosA$$

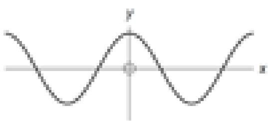


	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
Sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	1
Tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	---	0

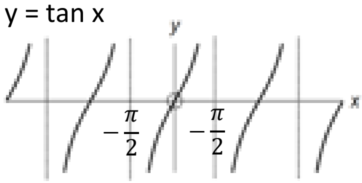
y = sin x



y = cos x

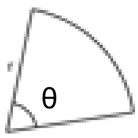


y = tan x



a) Radians 2π radians = 360° π radians = 180°

- You MUST work in radians if you are integrating or differentiating trig functions
- For an angle at the centre of a sector of θ **radians**



Arc Length = rθ
Area of the sector = ½r^2θ

b) Small angle approximations (θ in radians)

$$sin\theta \approx \theta \qquad cos\theta \approx 1 - \frac{\theta^2}{2} \qquad tan\theta \approx \theta$$

When θ is small show that $\frac{cos\theta}{sin\theta}$ can be written as $\frac{2-\theta^2}{2\theta}$

$$\left(1 - \frac{\theta^2}{2}\right) \div \theta$$

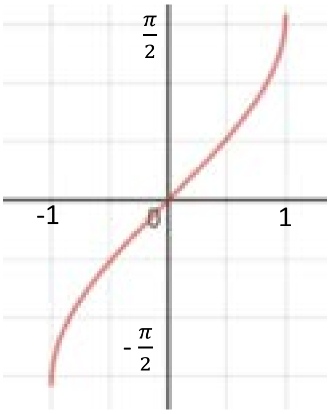
$$= \frac{2-\theta^2}{2} \div \theta$$

$$= \frac{2-\theta^2}{2\theta}$$

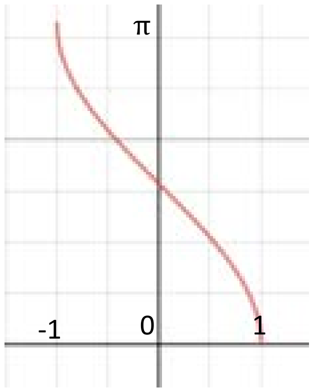
c) <u>Inverse Functions</u> ($\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$)

By definition a function must be one-to-one which leads to restricted domains for the inverse trig functions

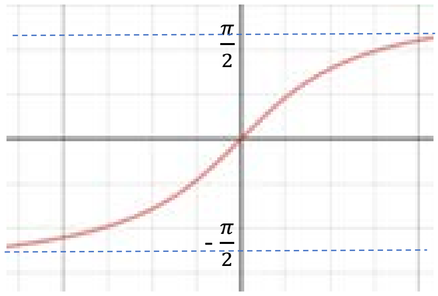
y = $\sin^{-1}$x (arcsin x)
Domain : $-1 \le x \le 1$



y = $\cos^{-1}$x (arccos x)
Domain : $-1 \le x \le 1$



y = $\tan^{-1}$x (arctan x)
Domain $x \in \mathbb{R}$



d) <u>Reciprocal Trig Functions and identities</u> (derived from $sin^2x + cos^2x = 1$)

$\sec x = \frac{1}{\cos x}$ $\qquad$ $cosec\ x = \frac{1}{\sin x}$ $\qquad$ $cot\ x = \frac{1}{\tan x}$ $\left(\frac{\cos x}{\sin x}\right)$

$1 + tan^2x = sec^2x$ $\qquad$ $1 + cot^2x = cosec^2x$

Solve for 0° < θ < 360° the equation $sec^4\theta - tan^4\theta = 2$
$sec^2x = 1 + tan^2x$
$sec^4x = 1 + 2tan^2x + tan^4x$ $\qquad$ $1 + 2tan^2x + tan^4x - tan^4x = 2$
$2tan^2x = 1$ $\qquad$ $\tan x = \pm\sqrt{\frac{1}{2}}$ $\qquad$ x = 35.3°, 145°, 215°,325° $\quad$ 3 s.f.

e) <u>Double angle and addition formulae</u>

The addition formulae are given the formula booklet
Make sure you can use these to derive :

DOUBLE ANGLE FORMULAE

sin 2A = 2 sinAcosA
cos 2A = $\cos^2$A – $\sin^2$A
= $2\cos^2$A – 1
= 1 - $2\sin^2$A
Tan 2A = $\frac{2\tan A}{1 - tan^2A}$

$\sin(A \pm B) = sinA\ cosB \pm cosAsinB$
$\cos(A \pm B) = cosA\ cosB \mp sinAsinB$
$\tan(A \pm B) = \dfrac{tanA \pm tanB}{1 \mp tanAtanB}$

- Useful to solve equations
- cos 2A often used to integrate trig functions involving $\sin^2$x or $\cos^2$x

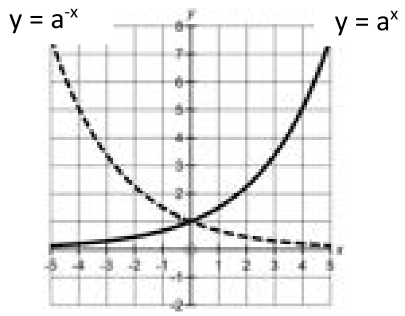
EXPRESSING IN THE FORM $\boldsymbol{rsin(\theta \pm \alpha)}$ and $\boldsymbol{rcos(\theta \pm \alpha)}$

- Useful in solving equations asin θ + bcos θ = 0
- Useful in finding minimum/maximum values of $acos\theta \pm bsin\theta$ and $asin\theta \pm bcos\theta$

Find the maximum value of the expression 2sinx + 3cosx and the value of x where this occurs (x < 180°)
2sinx + 3cosx = Rsin(x + α) $\quad$ (Rsin x cos α + Rcos x sin α) $\quad$ rcos α = 2 $\quad$ rsin α = 3
R = $\sqrt{2^2 + 3^2}$ $\qquad$ tanα = $\frac{3}{2}$
= $\sqrt{13}$ $\qquad$ α = 56.3° $\qquad$ 2sinx + 3cosx = $\sqrt{13}\sin(x + 56.3°)$
Max value = $\sqrt{13}$ occurs when $\sin(x + 56.3°)$ = 1
X = 33.7°

5 LOGARITHMS AND EXPONENTIALS

- A function of the form $y = a^x$ is an exponential function
- The graph of $y = a^x$ is positive for all values of x and passes through (0,1)
- A logarithm is the inverse of an exponential function
  $y = a^x$ $x = \log_a y$



Logarithms – rules to learn

$\log_a a = 1$ $\log_a 1 = 0$ $\log_a a^x = x$ $a^{\log_a x} = x$

$\log_a m + \log_a n = \log_a mn$ $\log_a m - \log_a n = \log_a \left(\frac{m}{n}\right)$ $k\log_a m = \log_a m^k$

Write the following in the form alog 2 where a is an integer 3log 2 + 2log 4 – ½log16

Method 1 : $\log 8 + \log 16 - \log 4 = \log\left(\frac{8\times16}{4}\right) = \log 32 = 5\log 2$

Method 2 : $3\log 2 + 4\log 2 - 2\log 2 = 5\log 2$

An equation of the form $a^x = b$ can be solved by taking logs of both sides

a) <u>MODELLING CURVES</u>
Exponential relationships can be changed to a linear form y = mx + c allowing the constants m and c to be 'estimated' from a graph of plotted data

$\mathbf{y = Ax^n}$ $\log y = \log(Ax^n)$ $\log y = n\log x + \log A$
$y \quad = mx \quad + \quad c$

Plot log y against log x. n is the gradient of the line and log A is the y axis intercept

$\mathbf{y = Ab^x}$ $\log y = \log(Ab^x)$ $\log y = x\log b + \log A$
$y \quad = mx \quad + \quad c$

Plot log y against x. log b is the gradient of the line and log A is the y axis intercept

V and x are connected by the equation $\mathbf{V = ax^b}$

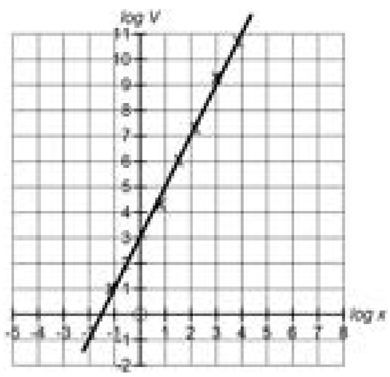
The equation is reduced to linear form by taking logs

$\log V = b\log x + \log a$

(y = mx + c) (log V plotted against log x)

From the graph b = 2

$\log a = 3 \quad a = 10^3$

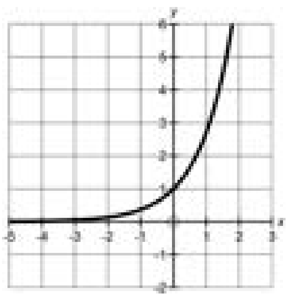


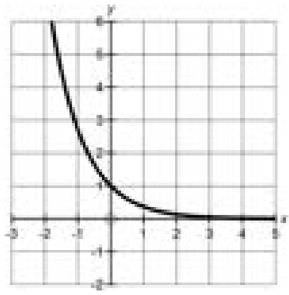
Gradient = 2

Intercept = 3

b) The exponential function y = e^x

Exponential Growth y = e^x **Exponential Decay** y = e^{-x}





The inverse of y = e^x is the **natural logarithm** denoted by **ln x**

Solve $2e^{x-2} = 6$ leaving your answer in exact form

$$e^{x-2} = 3$$
$$\ln(e^{x-2}) = \ln 3$$
$$x - 2 = \ln 3$$
$$x = \ln 3 + 2$$

The rate of growth/decay to find the 'rate of change' you need to differentiate to find the gradient

LEARN THIS

$$y = Ae^{kx} \qquad \frac{dy}{dx} = Ake^{kx}$$

The number of bacteria P in a culture is modelled by P = 600 + $5e^{0.2t}$ where t is the time in hours from the start of the experiment. Calculate the rate of growth after 5 hours

P = 600 + $15e^{0.2t}$ $\frac{dP}{dt} = 3e^{0.2t}$

t = 5 $\frac{dP}{dt} = 3e^{0.2\times 5}$

= 8.2 bacteria per hour

6 DIFFERENTIATION

- The gradient is denoted by $\frac{dy}{dx}$ if y is given as a function of x
- The gradient is denoted by f'(x) is the function is given as f(x)

LEARN THESE

$y = x^n \quad \frac{dy}{dx} = nx^{n-1}$ $\qquad y = ax^n \quad \frac{dy}{dx} = nax^{n-1}$ $\qquad y = a \quad \frac{dy}{dx} = 0$

$y = e^{kx} \quad \frac{dy}{dx} = ke^x$ $\qquad y = lnx \quad \frac{dy}{dx} = \frac{1}{x}$ $\qquad y = a^{kx} \quad \frac{dy}{dx} = (klna)a^{kx}$

$y = \sin kx \quad \frac{dy}{dx} = kcos\, kx$ $\qquad y = \cos kx \quad \frac{dy}{dx} = -ksin\, kx$ $\qquad y = \tan kx \quad \frac{dy}{dx} = ksec^2kx$

a) Methods of differentiation

CHAIN RULE for differentiating **y = fg(x)** y = f(u) where u = g(x) $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

PRODUCT RULE for differentiating **y = f(x)g(x)** $\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$

QUOTIENT RULE for differentiating **y =** $\frac{f(x)}{g(x)}$ $\quad \frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

PARAMETRIC EQUATIONS y = f(t) x = g(t) $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

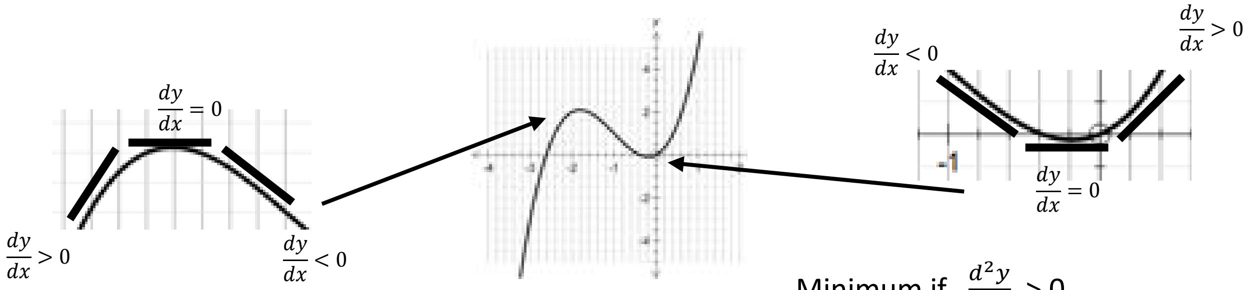
IMPLICIT DIFFERENTIATION- take care as you may need to use the product rule too (xy^2, xy, ysinx)

$$\frac{d[f(y)]}{dx} = \frac{d[f(y)]}{dy} \times \frac{dy}{dx}$$

b) <u>Stationary (Turning) Points</u>

- The points where $\frac{dy}{dx} = 0$ are stationary points (turning points/points of inflection) of a graph
- The nature of the turning points can be found by:

Maximum point **Minimum Point**



Minimum if $\frac{d^2y}{dx^2} > 0$

Maximum if $\frac{d^2y}{dx^2} < 0$

Find and determine the nature of the stationary points of the curve $y = 2x^3 - 3x^2 + 18$ $\frac{dy}{dx} = 6x^2 - 6x$ $\frac{dy}{dx} = 0$ at a stationary point $6x(x - 1) = 0$ Turning points at (0, 18) and (1,17) $\frac{d^2y}{dx^2} = 12x - 6$ $x = 0$ $\frac{d^2y}{dx^2} < 0$ (0,18) is a maximum $x = 1$ $\frac{d^2y}{dx^2} > 0$ (1,17) is a minimum

Points of inflection occur when $\frac{d^2y}{dx^2} = 0$ $(f''(x) = 0)$

………but $\frac{d^2y}{dx^2} = 0$ could also indicate a min or max point

Convex curve : $\frac{d^2y}{dx^2} > 0$ for all values of x in the 'convex section of the curve'

Concave curve : $\frac{d^2y}{dx^2} < 0$ for all values of x in the 'concave section of the curve'

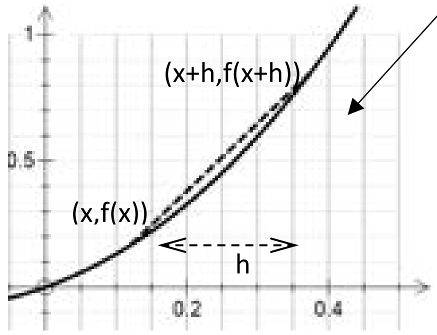
c) <u>Using Differentiation</u>

Tangents and Normals

The gradient of a curve at a given point = gradient of the tangent to the curve at that point
The gradient of the **normal** is perpendicular to the gradient of the tangent that point

Find the equation of the normal to the curve $y = 8x - x^2$ at the point (2,12) $\frac{dy}{dx} = 8 - 2x$ Gradient of tangent at (2,12) = 8 – 4 = 4 Gradient of the normal = - ¼ (y - 12) = -¼ (x – 2) 4y + x = 50

d) Differentiation from first principles



As h approaches zero the gradient of the chord gets closer to being the gradient of the tangent at the point

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

Find from first principles the derivative of $x^3 - 2x + 3$

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{(x+h)^3-2(x+h)+3-(x^3-2x+3)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{x^3+3x^2h+3xh^2+h^3-2x-2h+3-x^3+2x-3)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{3x^2h+3xh^2+h^3-2h}{h}\right)$$
$$= \lim_{h\to 0}(3x^2+3xh+h^2-2)$$
$$= 3x^2 - 2$$

7 INTEGRATION

Integration is the reverse of differentiation

LEARN THESE

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ (c is the constant of integration)

$\int e^{kx}dx = \frac{1}{k}e^{kx} + c$ $\int \frac{1}{x}dx = lnx + c$

$\int \sin kx\,dx = -\frac{1}{k}\cos kx + c$ $\int \cos kx\,dx = \frac{1}{k}\sin kx + c$

a) Methods of Integration

INTEGRATION BY SUBSTITUTION

Transforming a complex integral into a simpler integral using 'u = ' and integrating with respect to u

$\int x\sqrt{1-x^2}\,dx$

Let u = 1 - x^2 $\frac{du}{dx}$ = -2x so dx = $\frac{du}{-2x}$

$$\int x\sqrt{1-x^2}\,dx = \int x\sqrt{u}\,\frac{du}{-2x}$$
$$= -\frac{1}{2}\int u^{\frac{1}{2}}\,du$$
$$= -\frac{1}{3}u^{\frac{3}{2}} + c$$
$$= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c$$

If it is a definite integral it is often easier to calculate the limits in terms of u and substitute these in after integrating

Look for integrals of the form

$\int e^{ax+b}dx$ $\int \cos(ax+b)\,dx$ $\int \frac{1}{ax+b}dx$

Look out for integrals of the form

$$\int f'(x)[f(x)]^n = \frac{1}{n+1}[f(x)]^{n+1} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

INTEGRATION BY PARTS

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Take care in defining u and $\frac{dv}{dx}$

$\int xe^{2x}\,dx \qquad u = x \quad \frac{dv}{dx} = e^{2x}$

$\int x ln\,dx \qquad u = lnx \quad \frac{dv}{dx} = x$

$\int lnx\,dx \quad u = \ln x \quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x} \quad v = x$

$\int lnx\,dx = xlnx - \int x\frac{1}{x}\,dx$

$= xlnx - x + c$

PARAMETRIC INTEGRATION

To find the area under a curve defined parametrically use area = $\int y\frac{dx}{dt}dt$

Remember that the limits of the integral must be in terms of t

A curve is defined parametrically by $x = t - 1 \quad y = \frac{4}{t}$ Calculate the area of the region included by the line x= 2, the x-axis and the y-axis.

x = 2 t = 3 x = 0 t = 1

$\frac{dx}{dt} = 1 \qquad \int_1^3 \frac{4}{t}\,dt = [4lnt]_1^3$

$= 4ln3 - 4ln1$

= 4ln 3

b) AREA UNDER A CURVE

The area under a graph can be approximated using rectangle of height y and width dx. The limit as the number of rectangles increases is equal to the definite integral

$$\lim_{n\to\infty}\sum_{i=1}^{n} y_i\delta x = \int_a^b y\,dx$$

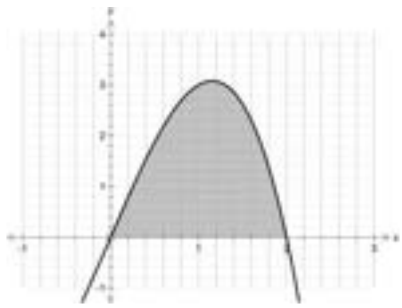
Calculate the area under the graph y = 4x – x³ between x = 0 and x = 2

$\int_0^2 4x - x^3 dx$

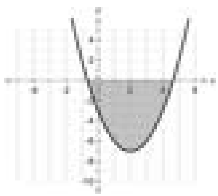
$= \left[2x^2 - \frac{x^4}{4}\right]_0^2$

$= (8-4) - (0-0)$

$= 4$



For an area below the x-axis the integral will result in a **negative value**

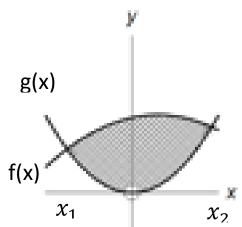


c) AREA BETWEEN 2 CURVES

If no limits are given you need to identify the x coordinates of the points where the curve intersect

Determine which function is 'above' the other

$\int_{x_1}^{x_2}[f(x) - g(x)]dx$



d) SOLUTION OF DIFFERENTIAL EQUATIONS

Separating the variables

If you are given the coordinates of a point on the curve a particular solution can be found if not a general solution is needed

Find the general solution for the differential equation

$y\frac{dy}{dx} = xy^2 + 3x$

$y\frac{dy}{dx} = x(y^2 + 3)$

$\int \frac{y}{y^2+3}\,dy = \int x\,dx$

$\frac{1}{2}\ln|y^2 + 3| = \frac{1}{2}x^2 + c$

8 NUMERICAL METHODS

a) CHANGE OF SIGN – locating a root

For an equations f(x) = 0 , if $f(x_1)$ and $f(x_2)$ have opposite signs and f(x) is a **continuous function** between x_1 and x_2 then a root of the equation lies in the interval $x_1 < x < x_2$

b) STAIRCASE and COWBEB DIAGRAMS

If an **iterative formula** (recurrence relation) of the form $x_{n+1}=f(x_n)$ converges to a limit, the value of the limit is the x-coordinate of the point of intersection of the graphs y = f(x) and y = x

The limit is the solution of the equation f(x) = x

A staircase or cobweb diagram based on the graphs y = f(x) and y = x shows the convergence

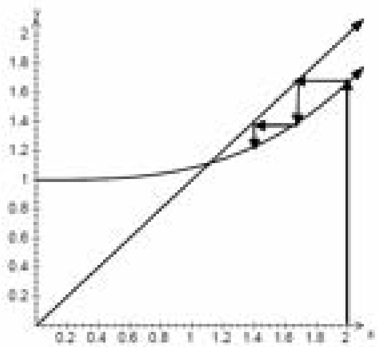
e.g Solve the equation $x^3 - 12x + 12 = 0$

First we will write it in the form x = f(x)

$x^3 + 12 = 12x \Rightarrow \frac{x^3}{12} + 1 = x$

Plotting the graphs $y = \frac{x^3}{12} + 1$ *and* $y = x$

the solution is the point of intersection of the two graphs.



We can confirm that there is a point of intersection between x = 1 and x = 2 by a change of sign the values are substituted.

Substituting x = 2 into $y = \frac{x^3}{12} + 1$ *gives y = 1.66… (shown on the diagram)*

Substituting x = 1.66…. y = 1.38…

Repeating this the values converge to 1.1157

The solution of $x^3 - 12x + 12 = 0$ is x = 1.1157

Use you **ANS** button on your calculator

c) NEWTON-RAPHSON iteration $f(x) = 0 \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

The equation $e^{-2x} - 0.5x = 0$ has a root close to 0.5. Using 0.5 as the first approximation use the Newton-Raphson you find the next approximation

$x_1 = 0.5 \quad f(0.5) = e^{-1} - 0.25$

$f'(x) = -2e^{-2x} - 0.5 \quad f'(0.5) = -2e^{-1} - 0.5$

$x_2 = 0.5 - \frac{e^{-1} - 0.25}{-2e^{-1} - 0.5} \qquad x_2 = 0.595$

Limitations of the Newton-Raphson method

As the method uses the tangent to the curve, if the starting value is a stationary point or close to a stationary point (min, max or inflection) the method does not work

d) <u>APPROXIMATING THE AREA UNDER A CURVE</u>

TRAPEZIUM RULE – given in the formula book but make sure you know how to use it!

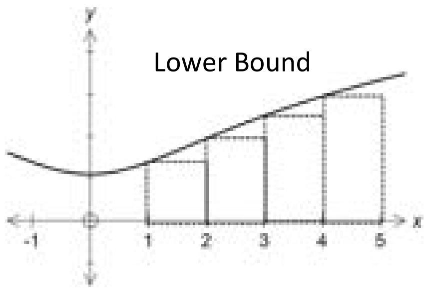
The trapezium rule gives an approximation of the area under a graph

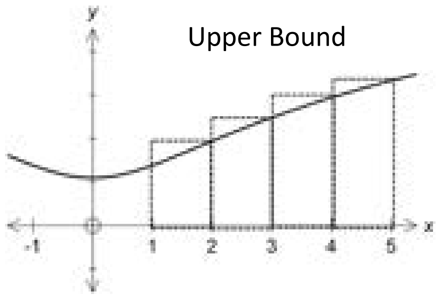
$$\int_a^b y\,dx \approx \frac{1}{2}h[(y_{0+}y_n) + 2(y_1 + y_2 + \ldots y_{n-1})] \quad \text{where } h = \frac{b-a}{n}$$

An easy way to calculate the y values is to use the TABLE function on a calculator – make sure you list the values in the formula (or a table) to show your method

- The rule will **underestimate** the area when the curve is **concave**
- The rule will **overestimate** the area when the curve is **convex**

UPPER and LOWER bounds - Area estimated using the area of rectangles
For the function shown below if the left hand 'heights' are used the total area is a Lower Bound – the rectangles calculated using the right hand heights the area results in the Upper Bound





9 VECTORS

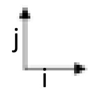
A vector has two properties **magnitude** (size) and **direction**

a) NOTATION

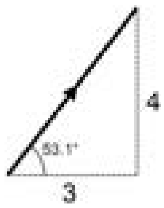
Vectors can be written as

a = $\binom{3}{4}$

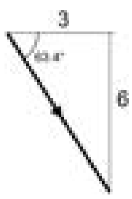
a = 3i + 4j where i and j perpendicular unit vectors (magnitude 1)



Magnitude-direction form (5, 53.1º) also known as **polar** form
The direction is the angle the vector makes with the **positive x axis**



Express the vector **p** = 3i – 6j in polar form



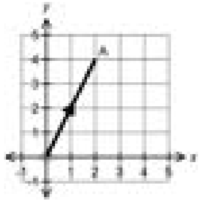
|**p**| = $\sqrt{3^2 + (-6)^2}$

= $3\sqrt{5}$

p = ($3\sqrt{5}$, 63.4º)

The **Magnitude** of vector **a** is denoted by |**a**| and can be found using Pythagoras |**a**| = $\sqrt{3^2 + 4^2}$
A **Unit Vector** is a vector which has magnitude 1

A **position vector** is a vector that starts at the origin (it has a fixed position)



$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $\quad 2i + 4j$

b) <u>ARITHMETIC WITH VECTORS</u>

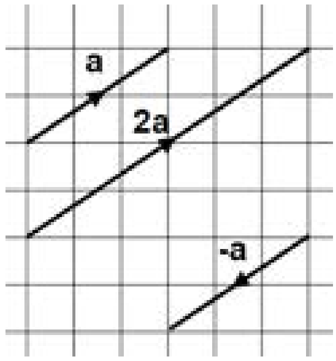
Multiplying by a scalar (number)

a = $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ 3i + 2j

2**a** = 2$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ = $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$ 6i + 4j

a and **2a** are **parallel** vectors

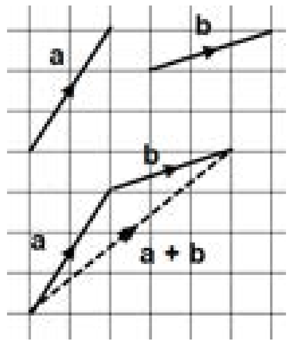
Multiplying by -1 reverses the direction of the



Addition of vectors

a = $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ **b** = $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

a + b = $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ = $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$

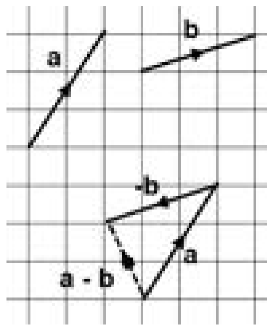


Subtraction of vectors

a = $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ **b** = $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

a - b = $\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ = $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

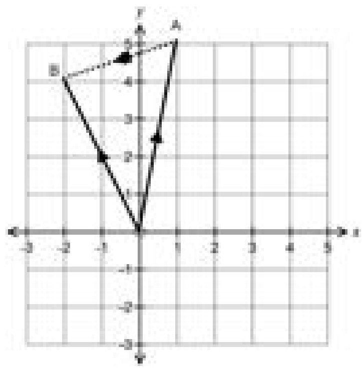
This is really **a + -b**



A and B have the coordinates (1,5) and (-2,4).

a) Write down the position vectors of A and B

$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ $\quad \overrightarrow{OB} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$



b) Write down the vector of the line segment joining A to B

$\overrightarrow{\boldsymbol{AB}} = -\overrightarrow{\boldsymbol{OA}} + \overrightarrow{\boldsymbol{OB}} \quad \boldsymbol{or} \quad \overrightarrow{\boldsymbol{OB}} - \overrightarrow{\boldsymbol{OA}}$

$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

Collinear - vectors in 2D and 3D can be used to show that 3 or more points are collinear (lie on a straight line)

Show that A(3,1,2) B(7,4,5) and C(19,13,14)

$\overrightarrow{AB} = 4i + 3j + 3k \quad \overrightarrow{BC} = 12i + 9j + 9k$

$\overrightarrow{BC} = \overrightarrow{3AB}$ $\quad \overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel vectors sharing a common point B and are therefore collinear

10 PROOF

Notation *If* $x = 3$ *then* $x^2 = 9$

$\Rightarrow$ $x = 3 \Rightarrow x^2 = 9$

$x = 3$ is a condition for $x^2 = 9$

$\Leftarrow$ $x = 3 \Leftarrow x^2 = 9$ is **not true** as x could = - 3

$\Leftrightarrow$ $x + 1 = 3 \Leftrightarrow x = 2$

a) Proof by deduction – statement proved using known mathematical principles

Useful expressions : 2n (an even number) 2n + 1 (an odd number)

Prove that the difference between the squares of any consecutive even numbers is a multiple of 4
Consecutive even numbers 2n, 2n + 2
$(2n + 2)^2 - (2n)^2$
$4n^2 + 8n + 4 - 4n^2$
$=8n + 4$
$=4(2n + 1)$ a multiple of 4

b) Proof by exhaustion – showing that a statement is true for every possible case or value

Prove that $(n + 2)^3 \geq 3^{n-1}$ for $n \epsilon \mathrm{N}$, n<4
We need to show it is true for 1,2 and 3
n = 1 $27 \geq 1$
n = 2 $64 \geq 3$
n = 3 $125 \geq 9$
True for all possible values hence proof that the statement is true by exhaustion

c) Disproof by counter example – finding an example that shows the statement is false.

Find a **counter example** for the statement
'2n + 4 is a multiple of 4'
n = 2 4 + 4 = 8 a multiple of 4
n = 3 6 + 4 = 10 NOT a multiple of 4

d) Proof by contradiction - assume first that the statement is not true and then show that this is not possible

Prove that for all integers n, if $n^3 + 5$ is odd then n is even

Assume that $n^3 + 5$ is odd and n is odd
Let $n^3 + 5 = 2k + 1$ and let $n = 2m + 1$ (k and m integers)

$2k + 1 = (2m + 1)^3 + 5$
$2k + 1 = 8m^3 + 12m^2 + 6m + 6$
$2k = 8m^3 + 12m^2 + 6m + 5$
$2k = 2(4m^3 + 6m^2 + 3m) + 5$
$k = (4m^3 + 6m^2 + 3m) + \frac{5}{2}$
$4m^3 + 6m^2 + 3m$ has an integer value leaving k as a non-integer value (contradicting assumptions)

Our initial assumption that when that $n^3 + 5$ is odd then n is odd is false so n must be even

AS LEVEL
REVISION QUESTIONS

1 Complete the table

Question	$f(x) = \frac{1}{2}(1 - \frac{1}{\sqrt{x}})$	$f(x) = (\frac{1}{x} + 1)^2$	$f(x) = (\frac{1}{x} + x)^2$	$f(x) = \frac{1}{x}(x - \frac{1}{x})$	$f(x) = \frac{1}{\sqrt{x}}(x - \frac{1}{\sqrt{x}})$	$f(x) = (\frac{1}{\sqrt{x}} + \sqrt{x})^2$
Brackets Expanded	$f(x) = \frac{1}{2} - \frac{1}{2\sqrt{x}}$					
Prepared For Differentiation	$f(x) = \frac{1}{2} - \frac{1}{2}x^{\frac{-1}{2}}$					
Differentiated	$f'(x) = \frac{1}{4}x^{\frac{-3}{2}}$					
Tidied Up	$f'(x) = \frac{1}{4x^{\frac{3}{2}}}$ $f'(x) = \frac{1}{4\sqrt{x^3}}$					

2 Complete the table

Question	$f(x) = \frac{x^2 - 2x + 1}{\sqrt{x}}$	$f(x) = \frac{x^2 + 2x + 1}{x}$	$f(x) = \frac{3x^2 + 4x + 1}{4x^2}$	$f(x) = \frac{x + 1}{\sqrt{x}}$	$f(x) = \frac{(1 + 2x)^2}{\sqrt{x}}$	$f(x) = \frac{x^2 - 4x}{x\sqrt{x}}$
Fraction Broken Up	$f(x) = \frac{x^2}{\sqrt{x}} - \frac{2x}{\sqrt{x}} + \frac{1}{\sqrt{x}}$					
Prepared For Differentiation	$f(x) = \frac{x^2}{x^{\frac{1}{2}}} - \frac{2x}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}}$ $f(x) = x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + x^{\frac{-1}{2}}$					
Differentiated	$f'(x) = \frac{3}{2}x^{\frac{1}{2}} - x^{\frac{-1}{2}} - \frac{1}{2}x^{\frac{-3}{2}}$					
Tidied Up	$f'(x) = \frac{3}{2}\sqrt{x} - \frac{1}{x^{\frac{1}{2}}} - \frac{1}{2x^{\frac{3}{2}}}$ $f'(x) = \frac{3}{2}\sqrt{x} - \frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$					

1. $\int x\,\mathrm{d}x$
2. $\int 3\,\mathrm{d}x$
3. $\int 3x\,\mathrm{d}x$
4. $\int \mathrm{d}x$
5. $\int \frac{x^4}{4}\mathrm{d}x$
6. $\int \frac{2x^3}{7}\mathrm{d}x$
7. $\int \frac{1}{x^2}\mathrm{d}x$
8. $\int \frac{3}{x^2}\mathrm{d}x$
9. $\int \frac{1}{3x^2}\mathrm{d}x$
10. $\int \frac{\mathrm{d}x}{2x^5}$
11. $\int \frac{4}{3x^5}\mathrm{d}x$
12. $\int \sqrt{x}\,\mathrm{d}x$
13. $\int 3\sqrt{x}\,\mathrm{d}x$
14. $\int \sqrt[3]{x}\,\mathrm{d}x$
15. $\int 4\sqrt[5]{x^2}\,\mathrm{d}x$
16. $\int \frac{\sqrt[6]{x}}{2}\mathrm{d}x$
17. $\int \frac{2}{\sqrt[3]{x}}\mathrm{d}x$
18. $\int \frac{\mathrm{d}x}{2\sqrt[4]{x^3}}$
19. $\int (5-3x)^2\mathrm{d}x$
20. $\int (1+\frac{1}{x^2})^2\mathrm{d}x$
21. $\int 3\sqrt{x}(1-\sqrt{x})\mathrm{d}x$
22. $\int \frac{4+x^4}{x^3}\mathrm{d}x$
23. $\int \frac{4+x^3}{x^5}\mathrm{d}x$
24. $\int \frac{x+5x^7}{x^3}\mathrm{d}x$
25. $\int \frac{2+\sqrt{x}}{\sqrt{x}}\mathrm{d}x$
26. $\int \frac{6x^2+x-7}{\sqrt{x}}\mathrm{d}x$
27. $\int \frac{6x^2+x-7}{x\sqrt{x}}\mathrm{d}x$
28. This exercise did not include: $\int \frac{1}{x}\mathrm{d}x$ Investigate this integral.

    Do not Google the answer! Desmos, however, might be useful in your investigating.

29. Create another 3 integration problems of your own. One which is similar to the questions you found easy, one which is similar to the problems you found to be of medium difficulty and one which is similar to the ones you found hard!

    Now create a visual guide on how to calculate each of the three types of integral.

Year 13 – AS Revision Sheet 1

Section A

1) Find the value of $125^{\frac{-2}{3}}$ **2**

2) Simplify $x^7(16x^{20})^{\frac{1}{4}}$ **2**

3) Given that $(2+\sqrt{5})(a+b\sqrt{5}) = (4+3\sqrt{5})$ calculate the values of a and b. **4**

4) Solve $3^{3x-4} = 9^x$ **3**

5) Given that $x^2 - 10x - 3 \equiv (x+a)^2 + b$ find the values of a and b.

Hence, or otherwise, solve $x^2 - 10x - 3 = 0$ **4**

6) a) On a coordinate grid (x and y axes running from −6 to 6), shade the region comprising all points whose coordinates satisfy the inequalities

$y \leq 2x + 5$, $2y + x \leq 6$ and $y \geq 2$ **3**

b) Work out the area of the shaded region. **5**

7) Write $\frac{4\sqrt{x}+5}{x}$ in the form $4x^m + 5x^n$ clearly stating the values of m and n. **2**

8) The circle C has equation $x^2 + y^2 + 4x - 2y - 11 = 0$. Find

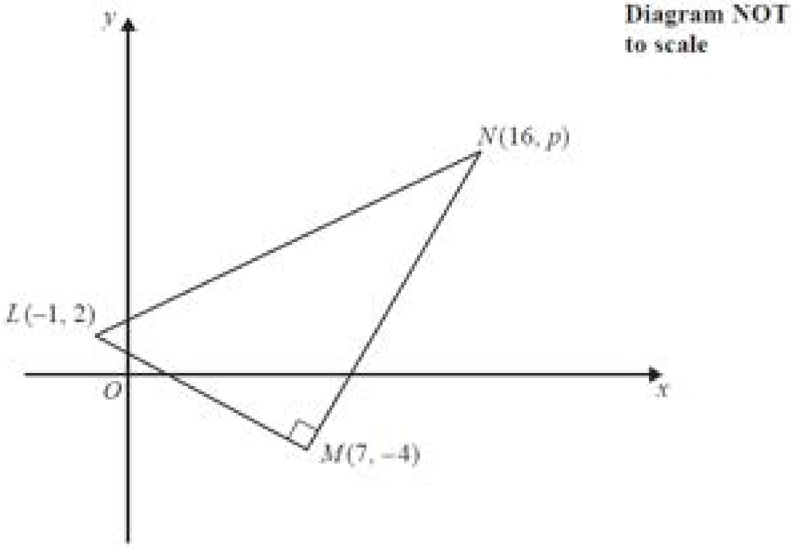
a) the coordinates of the centre of C, **2**

b) the radius of C, **2**

c) the coordinates of the points where C crosses the y-axis, giving your answers as simplified surds. **4**

Year 13 – AS Revision Sheet 1

Section A



9) The diagram above shows a right angled triangle *LMN*.

The points *L* and *M* have coordinates (–1, 2) and (7, –4) respectively.

(*a*) Find an equation for the straight line passing through the points *L* and *M*.

Give your answer in the form $ax + by + c = 0$, where a, b and c are integers. **(4)**

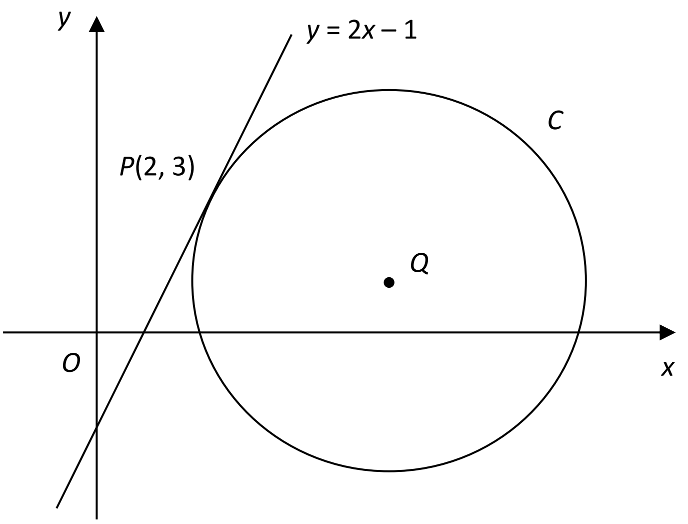
Given that the coordinates of point *N* are (16, *p*), where *p* is a constant, and angle $LMN = 90°$,

(*b*) find the value of *p*. **(3)**

Given that there is a point *K* such that the points *L*, *M*, *N*, and *K* form a rectangle,

(*c*) find the *y* coordinate of *K*. **(2)**

10) The line $y = 2x - 1$ is a tangent to the circle *C*, touching *C* at the point *P* (2, 3) as shown. Point *Q* is the centre of the circle.



a) Find the equation of the line joining point *P* to the point *Q*.

(3)

b) If the *x*-coordinate of point *Q* is 6 find the equation of the circle *C*.

(5)

Year 13 – AS Revision Sheet 2

Section A

1) Find the set of values of x for which

(*a*) $3(x - 2) < 8 - 2x$, **(2)**

(*b*) $(2x - 7)(1 + x) < 0$, **(3)**

(*c*) both $3(x - 2) < 8 - 2x$ **and** $(2x - 7)(1 + x) < 0$. **(1)**

2) The point $P(1, a)$ lies on the curve with equation $y = (x + 1)^2(2 - x)$.

(*a*) Find the value of a. **(1)**

(*b*) Sketch the curves with the following equations:

(i) $y = (x + 1)^2(2 - x)$,

(ii) $y = \dfrac{2}{x}$.

On your diagram show clearly the coordinates of any points at which the curves meet the axes. **(5)**

(*c*) With reference to your diagram in part (*b*), state the number of real solutions to the equation

$(x + 1)^2(2 - x) = \dfrac{2}{x}$. **(1)**

3) The curve C has equation $y = kx^3 - x^2 + x - 5$, where k is a constant.

(*a*) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. **(2)**

The point A with x-coordinate $-\frac{1}{2}$ lies on C. The tangent to C at A is parallel to the line with equation $2y - 7x + 1 = 0$.

Find

(*b*) the value of k, **(4)**

(*c*) the value of the y-coordinate of A. **(2)**

Year 13 – AS Revision Sheet 2

Section A

4) (*a*) Find the first 3 terms, in ascending powers of x, of the binomial expansion of

$$(3 + bx)^5$$

where b is a non-zero constant. Give each term in its simplest form. **(4)**

Given that, in this expansion, the coefficient of x^2 is twice the coefficient of x,

(*b*) find the value of b. **(2)**

5) (*a*) Find the positive value of x such that

$$\log_x 64 = 2.$$ **(2)**

(*b*) Solve for x

$$\log_2 (11 - 6x) = 2 \log_2 (x - 1) + 3.$$ **(6)**

6) (*a*) Find, to 3 significant figures, the value of x for which $5^x = 7$. **(2)**

(*b*) Solve the equation $5^{2x} - 12(5^x) + 35 = 0$. **(4)**

Year 2: A Level Mathematics	
<u>Algebra and Functions: Partial Fractions</u>	
Self-Assessment: Please identify areas in which you believe are your strong points and those you feel you need to improve on Provide evidence to support your assessment with reference to the content in this booklet.	
Strengths	**Areas for Improvement**

Partial fractions

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An algebraic fraction such as $\frac{3x+5}{2x^2-5x-3}$ can often be broken down into simpler parts called partial fractions. Specifically

$$\frac{3x+5}{2x^2-5x-3} = \frac{2}{x-3} - \frac{1}{2x+1}$$

In this unit we explain how this process is carried out.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- explain the meaning of the terms 'proper fraction' and 'improper fraction'
- express an algebraic fraction as the sum of its partial fractions

Contents

1. Introduction

An **algebraic fraction** is a fraction in which the numerator and denominator are both polynomial expressions. A **polynomial expression** is one where every term is a multiple of a power of x, such as

$$5x^4 + 6x^3 + 7x + 4$$

The **degree** of a polynomial is the power of the highest term in x. So in this case the degree is 4.

The number in front of x in each term is called its **coefficient**. So, the coefficient of x^4 is 5. The coefficient of x^3 is 6.

Now consider the following algebraic fractions:

$$\frac{x}{x^2+2} \qquad \frac{x^3+3}{x^4+x^2+1}$$

In both cases the numerator is a polynomial of lower degree than the denominator. We call these **proper fractions**

With other fractions the polynomial may be of higher degree in the numerator or it may be of the same degree, for example

$$\frac{x^4+x^2+x}{x^3+x+2} \qquad \frac{x+4}{x+3}$$

and these are called **improper fractions**.

Key Point

If the degree of the numerator is less than the degree of the denominator the fraction is said to be a **proper fraction**

If the degree of the numerator is greater than or equal to the degree of the denominator the fraction is said to be an **improper fraction**

2. Revision of adding and subtracting fractions

We now revise the process for adding and subtracting fractions. Consider

$$\frac{2}{x-3} - \frac{1}{2x+1}$$

In order to add these two fractions together, we need to find the lowest common denominator. In this particular case, it is $(x-3)(2x+1)$.

We write each fraction with this denominator.

$$\frac{2}{x-3} = \frac{2(2x+1)}{(x-3)(2x+1)} \qquad \text{and} \qquad \frac{1}{2x+1} = \frac{x-3}{(x-3)(2x+1)}$$

So

$$\frac{2}{x-3} - \frac{1}{2x+1} = \frac{2(2x+1)}{(x-3)(2x+1)} - \frac{x-3}{(x-3)(2x+1)}$$

The denominators are now the same so we can simply subtract the numerators and divide the result by the lowest common denominator to give

$$\frac{2}{x-3} - \frac{1}{2x+1} = \frac{4x+2-x+3}{(x-3)(2x+1)} = \frac{3x+5}{(x-3)(2x+1)}$$

Sometimes in mathematics we need to do this operation in reverse. In calculus, for instance, or when dealing with the binomial theorem, we sometimes need to split a fraction up into its component parts which are called **partial fractions**. We discuss how to do this in the following section.

Exercises 1

Use the rules for the addition and subtraction of fractions to simplify

a) $\dfrac{3}{x+1} + \dfrac{2}{x+3}$ b) $\dfrac{5}{x-2} - \dfrac{3}{x+2}$ c) $\dfrac{4}{2x+1} - \dfrac{2}{x+3}$ d) $\dfrac{1}{3x-1} - \dfrac{2}{6x+9}$

3. Expressing a fraction as the sum of its partial fractions

In the previous section we saw that

$$\frac{2}{x-3} - \frac{1}{2x+1} = \frac{3x+5}{(x-3)(2x+1)}$$

Suppose we start with $\dfrac{3x+5}{(x-3)(2x+1)}$. How can we get this back to its component parts ?

By inspection of the denominator we see that the component parts must have denominators of $x-3$ and $2x+1$ so we can write

$$\frac{3x+5}{(x-3)(2x+1)} = \frac{A}{x-3} + \frac{B}{2x+1}$$

where A and B are numbers. A and B cannot involve x or powers of x because otherwise the terms on the right would be improper fractions.

The next thing to do is to multiply both sides by the common denominator $(x-3)(2x+1)$. This gives

$$\frac{(3x+5)(x-3)(2x+1)}{(x-3)(2x+1)} = \frac{A(x-3)(2x+1)}{x-3} + \frac{B(x-3)(2x+1)}{2x+1}$$

Then cancelling the common factors from the numerators and denominators of each term gives

$$3x+5 = A(2x+1) + B(x-3)$$

Now this is an **identity**. This means that it is true for any values of x, and because of this we can substitute any values of x we choose into it. Observe that if we let $x = -\frac{1}{2}$ the first term on the right will become zero and hence A will disappear. If we let $x = 3$ the second term on the right will become zero and hence B will disappear.

If $x = -\frac{1}{2}$

$$\begin{aligned} -\frac{3}{2} + 5 &= B\left(-\frac{1}{2} - 3\right) \\ \frac{7}{2} &= -\frac{7}{2}B \end{aligned}$$

from which

$$B = -1$$

Now we want to try to find A.

If $x = 3$

$$14 = 7A$$

so that $A = 2$.
Putting these results together we have

$$\begin{aligned} \frac{3x+5}{(x-3)(2x+1)} &= \frac{A}{x-3} + \frac{B}{2x+1} \\ &= \frac{2}{x-3} - \frac{1}{2x+1} \end{aligned}$$

which is the sum that we started with, and we have now broken the fraction back into its component parts called **partial fractions**.

Example

Suppose we want to express $\frac{3x}{(x-1)(x+2)}$ as the sum of its partial fractions.

Observe that the factors in the denominator are $x - 1$ and $x + 2$ so we write

$$\frac{3x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

where A and B are numbers.

We multiply both sides by the common denominator $(x - 1)(x + 2)$:

$$3x = A(x + 2) + B(x - 1)$$

This time the special values that we shall choose are $x = -2$ because then the first term on the right will become zero and A will disappear, and $x = 1$ because then the second term on the right will become zero and B will disappear.

If $x = -2$

$$\begin{aligned} -6 &= -3B \\ B &= \frac{-6}{-3} \\ B &= 2 \end{aligned}$$

If $x = 1$

$$\begin{aligned} 3 &= 3A \\ A &= 1 \end{aligned}$$

Putting these results together we have

$$\frac{3x}{(x-1)(x+2)} = \frac{1}{x-1} + \frac{2}{x+2}$$

and we have expressed the given fraction in partial fractions.

Sometimes the denominator is more awkward as we shall see in the following section.

Exercises 2

Express the following as a sum of partial fractions

a) $\dfrac{2x-1}{(x+2)(x-3)}$ b) $\dfrac{2x+5}{(x-2)(x+1)}$ c) $\dfrac{3}{(x-1)(2x-1)}$ d) $\dfrac{1}{(x+4)(x-2)}$

4. Fractions where the denominator has a repeated factor

Consider the following example in which the denominator has a repeated factor $(x-1)^2$.

Example

Suppose we want to express $\dfrac{3x+1}{(x-1)^2(x+2)}$ as the sum of its partial fractions.

There are actually three possibilities for a denominator in the partial fractions: $x-1$, $x+2$ and also the possibility of $(x-1)^2$, so in this case we write

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

where A, B and C are numbers.

As before we multiply both sides by the denominator $(x-1)^2(x+2)$ to give

$$3x+1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \qquad (1)$$

Again we look for special values to substitute into this identity. If we let $x=1$ then the first and last terms on the right will be zero and A and C will disappear. If we let $x=-2$ the first and second terms will be zero and A and B will disappear.

If $x = 1$

$$4 = 3B \qquad \text{so that} \qquad B = \frac{4}{3}$$

If $x = -2$

$$-5 = 9C \qquad \text{so that} \qquad C = -\frac{5}{9}$$

We now need to find A. There is no special value of x that will eliminate B and C to give us A. We could use any value. We could use $x=0$. This will give us an equation in A, B and C. Since we already know B and C, this would give us A.

But here we shall demonstrate a different technique - one called *equating coefficients*. We take equation 1 and multiply-out the right-hand side, and then collect up like terms.

$$\begin{aligned}
3x+1 &= A(x-1)(x+2)+B(x+2)+C(x-1)^2 \\
&= A(x^2+x-2)+B(x+2)+C(x^2-2x+1) \\
&= (A+C)x^2+(A+B-2C)x+(-2A+2B+C)
\end{aligned}$$

This is an identity which is true for all values of x. On the left-hand side there are no terms involving x^2 whereas on the right we have $(A+C)x^2$. The only way this can be true is if

$$A+C=0$$

This is called **equating coefficients** of x^2. We already know that $C=-\frac{5}{9}$ so this means that $A=\frac{5}{9}$. We also already know that $B=\frac{4}{3}$. Putting these results together we have

$$\frac{3x+1}{(x-1)^2(x+2)}=\frac{5}{9(x-1)}+\frac{4}{3(x-1)^2}-\frac{5}{9(x+2)}$$

and the problem is solved.

Exercises 3

Express the following as a sum of partial fractions

a) $\dfrac{5x^2+17x+15}{(x+2)^2(x+1)}$ b) $\dfrac{x}{(x-3)^2(2x+1)}$ c) $\dfrac{x^2+1}{(x-1)^2(x+1)}$

5. Fractions in which the denominator has a quadratic term

Sometimes we come across fractions in which the denominator has a quadratic term which cannot be factorised. We will now learn how to deal with cases like this.

Example

Suppose we want to express

$$\frac{5x}{(x^2+x+1)(x-2)}$$

as the sum of its partial fractions.

Note that the two denominators of the partial fractions will be (x^2+x+1) and $(x-2)$. When the denominator contains a quadratic factor we have to consider the possibility that the numerator can contain a term in x. This is because if it did, the numerator would still be of lower degree than the denominator - this would still be a proper fraction. So we write

$$\frac{5x}{(x^2+x+1)(x-2)}=\frac{Ax+B}{x^2+x+1}+\frac{C}{x-2}$$

As before we multiply both sides by the denominator $(x^2+x+1)(x-2)$ to give

$$5x=(Ax+B)(x-2)+C(x^2+x+1)$$

One special value we could use is $x=2$ because this will make the first term on the right-hand side zero and so A and B will disappear.

If $x = 2$

$$10 = 7C \qquad \text{and so} \qquad C = \frac{10}{7}$$

Unfortunately there is no value we can substitute which will enable us to get rid of C so instead we use the technique of equating coefficients. We have

$$\begin{aligned} 5x &= (Ax + B)(x - 2) + C(x^2 + x + 1) \\ &= Ax^2 - 2Ax + Bx - 2B + Cx^2 + Cx + C \\ &= (A + C)x^2 + (-2A + B + C)x + (-2B + C) \end{aligned}$$

We still need to find A and B. There is no term involving x^2 on the left and so we can state that

$$A + C = 0$$

Since $C = \dfrac{10}{7}$ we have $A = -\dfrac{10}{7}$.

The left-hand side has no constant term and so

$$-2B + C = 0 \qquad \text{so that} \qquad B = \frac{C}{2}$$

But since $C = \dfrac{10}{7}$ then $B = \dfrac{5}{7}$. Putting all these results together we have

$$\begin{aligned} \frac{5x}{(x^2 + x + 1)(x - 2)} &= \frac{-\frac{10}{7}x + \frac{5}{7}}{x^2 + x + 1} + \frac{\frac{10}{7}}{x - 2} \\ &= \frac{-10x + 5}{7(x^2 + x + 1)} + \frac{10}{7(x - 2)} \\ &= \frac{5(-2x + 1)}{7(x^2 + x + 1)} + \frac{10}{7(x - 2)} \end{aligned}$$

Exercises 4

Express the following as a sum of partial fractions

a) $\dfrac{x^2 - 3x - 7}{(x^2 + x + 2)(2x - 1)}$ b) $\dfrac{13}{(2x + 3)(x^2 + 1)}$ c) $\dfrac{x}{(x^2 - x + 1)(3x - 2)}$

6. Dealing with improper fractions

So far we have only dealt with proper fractions, for which the numerator is of lower degree than the denominator. We now look at how to deal with improper fractions.

Consider the following example.

Example

Suppose we wish to express $\dfrac{4x^3 + 10x + 4}{x(2x + 1)}$ in partial fractions.

The numerator is of degree 3. The denominator is of degree 2. So this fraction is improper. This means that if we are going to divide the numerator by the denominator we are going to divide a term in x^3 by one in x^2, which gives rise to a term in x. Consequently we express the partial fractions in the form:

$$\frac{4x^3 + 10x + 4}{x(2x+1)} = Ax + B + \frac{C}{x} + \frac{D}{2x+1}$$

Multiplying both sides by the denominator $x(2x+1)$ gives

$$4x^3 + 10x + 4 = Ax^2(2x+1) + Bx(2x+1) + C(2x+1) + Dx$$

Note that by substituting the special value $x = 0$, all terms on the right except the third will be zero. If we use the special value $x = -\frac{1}{2}$ all terms on the right except the last one will be zero.

If $x = 0$

$$4 = C$$

If $x = -\dfrac{1}{2}$

$$\begin{aligned} -\frac{4}{8} - \frac{10}{2} + 4 &= -\frac{1}{2}D \\ -\frac{1}{2} - 5 + 4 &= -\frac{1}{2}D \\ -1\frac{1}{2} &= -\frac{1}{2}D \\ D &= 3 \end{aligned}$$

Special values will not give A or B so we shall have to equate coefficients.

$$\begin{aligned} 4x^3 + 10x + 4 &= Ax^2(2x+1) + Bx(2x+1) + C(2x+1) + Dx \\ &= 2Ax^3 + Ax^2 + 2Bx^2 + Bx + 2Cx + C + Dx \\ &= 2Ax^3 + (A+2B)x^2 + (B+2C+D)x + C \end{aligned}$$

Now look at the term in x^3.

$$2A = 4 \qquad \text{so that} \qquad A = 2$$

Now look at the term in x^2. There is no such term on the left. So

$$A + 2B = 0 \qquad \text{so that} \qquad A = -2B \qquad \text{so that} \qquad B = -\frac{2}{2} = -1$$

Putting all these results together gives

$$\frac{4x^3 + 10x + 4}{x(2x+1)} = 2x - 1 + \frac{4}{x} + \frac{3}{2x+1}$$

and the problem is solved.

Exercise 5

Express the following as a sum of powers of x and partial fractions

a) $\dfrac{x^3+1}{x^2+1}$ b) $\dfrac{2x^4+3x^2+1}{x^2+3x+2}$ c) $\dfrac{7x^2-1}{x+3}$

Answers

Exercise 1

a) $\dfrac{5x+11}{(x+1)(x+3)}$ b) $\dfrac{2x+16}{(x-2)(x+2)}$ c) $\dfrac{10}{(2x+1)(x+3)}$ d) $\dfrac{11}{(3x-1)(6x+9)}$

Exercise 2

a) $\dfrac{1}{x+2}+\dfrac{1}{x-3}$ b) $\dfrac{3}{x-2}-\dfrac{1}{x+1}$ c) $\dfrac{3}{x-1}-\dfrac{6}{2x-1}$ d) $\dfrac{1}{6(x-2)}-\dfrac{1}{6(x+4)}$

Exercise 3

a) $\dfrac{2}{x+2}-\dfrac{1}{(x+2)^2}+\dfrac{3}{x+1}$ b) $\dfrac{1}{49(x-3)}+\dfrac{3}{7(x-3)^2}-\dfrac{2}{49(2x+1)}$

c) $\dfrac{1}{2(x-1)}+\dfrac{1}{(x-1)^2}+\dfrac{1}{2(x+1)}$

Exercise 4

a) $\dfrac{2x+1}{x^2+x+2}-\dfrac{3}{2x-1}$ b) $\dfrac{4}{2x+3}-\dfrac{2x-3}{x^2+1}$ c) $\dfrac{-2x+3}{7(x^2-x+1)}+\dfrac{6}{7(3x-2)}$

Exercise 5

a) $x+\dfrac{-x+1}{x^2+1}$ b) $2x^2-6x+17+\dfrac{6}{x+1}-\dfrac{45}{x+2}$ c) $7x-21+\dfrac{62}{x+3}$

Year 2: A Level Mathematics	
<u>Coordinate Geometry: Parametric Equations</u>	
Self-Assessment: Please identify areas in which you believe are your strong points and those you feel you need to improve on Provide evidence to support your assessment with reference to the content in this booklet.	
Strengths	**Areas for Improvement**

Parametric Equations Cheat Sheet

So far, we have only looked at functions given in two variables, y and x. This is known as the cartesian equation of a curve. We can also define a curve using a different system, known as parametric equations.

We define the x and y coordinates separately, in terms of a third variable, t:

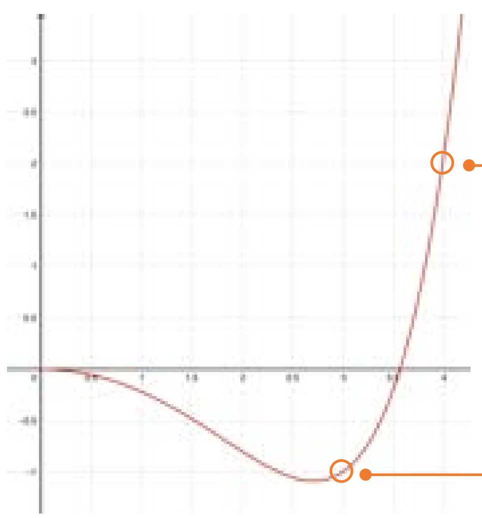
- $x = p(t)$
- $y = q(t)$

Each value of t defines a point on the curve.

To develop a better understanding of how this works, let's look at the following curve defined parametrically:

$x = 3\sqrt{t}, \quad y = t^3 - 2t, \quad 0 < t < 2$

This is the domain of the parameter t, which tells us the values of t which we can substitute into x and y to find points on the curve.



If we wanted to find the value of t at this point (4, 2) then using $x = 3\sqrt{t}$, we let $x = 4$: $4 = 3\sqrt{t}$, $\frac{4}{3} = \sqrt{t} \therefore t = \frac{16}{9}$

We could also let $y = 2$ and solve $2 = t^3 - 2t$, but this would be more difficult.

At $t = 1$, we have that $x = 3\sqrt{1} = 3$ and $y = (1)^2 - 2(1) = -1$
$\therefore$ the point $(3, -1)$ corresponds to when $t = 1$.

Converting between parametric and cartesian equations

To convert between parametric and cartesian equations, you must use substitution to eliminate the parameter. You also need to be able to relate the domain and range of a cartesian equation to its parametric counterpart. Remember that:

- The domain of $f(x)$ is the range of $p(t)$
- The range of $f(x)$ is the range of $q(t)$

Example 1: A curve has parametric equations. $x = \ln(4-t), \quad y = t - 2, \quad t < 3$
(a) Find the cartesian equation for the curve in the form $y = f(x)$.
(b) Find the domain and range of $f(x)$.

a) Using $x = \ln(4-t)$, we start by making t the subject:	$e^x = 4 - t$ $\therefore \quad t = 4 - e^x$
Substituting into y:	$y = (4 - e^x) - 2$ $\Rightarrow y = 2 - e^x$
b) We use the domain/range properties of parametric functions to deduce the domain and range of $f(x)$	The domain of $f(x)$ is the range of $\ln(4-t)$ for $t < 3$. By a sketch or otherwise, you can deduce this is $x > 0$. The range of $f(x)$ is the range of $t - 2$ for $t < 3$. This will be $y < 1$.

When the parametric equations involve trigonometric functions, you may need to use trigonometric identities to convert to cartesian form. Here are two examples showing how this is done in practice:

Example 2: A curve C has parametric equations
$x = \cot t, \quad y = cosec^2 t - 2, \quad 0 < t < \pi$

Find the cartesian equation of the curve in the form $y = f(x)$.

Using $1 + cot^2 x = cosec^2 x$	$x^2 = cot^2 t = cosec^2 t - 1$ $so\ x^2 + 1 = cosec^2 t$
Substituting into y:	$\Rightarrow y = x^2 + 1 - 2$ $\Rightarrow y = x^2 - 1$
Using $1 + cot^2 x = cosec^2 x$	$x^2 = cot^2 t = cosec^2 t - 1$ $so\ x^2 + 1 = cosec^2 t$

Example 3: A curve C has parametric equations
$x = 2cost, \quad y = \sin\left(t - \frac{\pi}{6}\right), \quad 0 < t < \pi$
Find a cartesian equation of the curve in the form $y = f(x)$, stating its domain.

a) We start by expanding y using the addition formulae from Chapter 7 of Pure Year 2:	$y = \sin(t)\, cos\left(\frac{\pi}{6}\right) - \cos(t)\, sin\left(\frac{\pi}{6}\right)$ $= \frac{\sqrt{3}}{2} sint - \frac{1}{2} cost$
Using the result from the previous step and substituting $x = 2cost$:	Since $x = 2cost$, $y = \frac{\sqrt{3}}{2} sint - \frac{1}{4}x$.
Now we need to substitute out $sint$ as it is the only remaining term with t in it. We can use the identity $sin^2 t + cos^2 t = 1$ to do this.	$\left(\frac{x}{2}\right) = cost \therefore \left(\frac{x}{2}\right)^2 = cos^2 t$ So $sin^2 t = 1 - \left(\frac{x}{2}\right)^2$ and $sint = \sqrt{1 - \left(\frac{x}{2}\right)^2}$
Substituting this new expression for $sint$ back into the expression from the 2nd step: We now have the required cartesian equation.	$\therefore y = \frac{\sqrt{3}}{2}\left(\sqrt{1 - \left(\frac{x}{2}\right)^2}\right) - \frac{1}{4}x$
We look at the range of the given parametric equation, $x = 2cost$, to find the domain of our cartesian equation. Alternatively, you could also substitute $t = 0$ and $t = \pi$ into $x = 2cost$ to find the domain.	The domain of this function is the range of $x = 2cost$ for $0 < t < \pi$. By a quick sketch of $x = 2cost$, we can see that x takes on all values between but not including -2 and 2, so our is $-2 < x < 2$

Sketching parametric equations

Parametric curves are usually more difficult to sketch than curves given in cartesian form.

- To plot parametric curves, we need to construct a table of values and use it to sketch the curve.

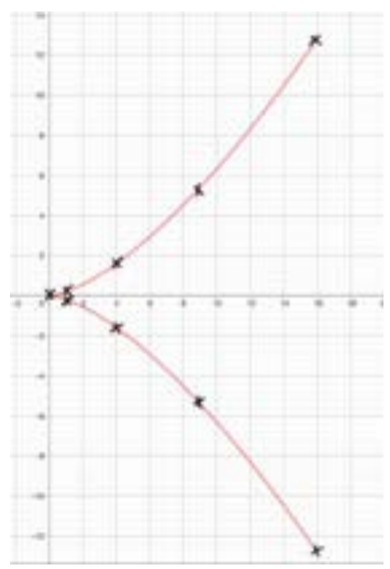
Example 4: A curve is given by the parametric equations $x = t^2$, $y = \frac{t^3}{5}$
Sketch the curve for $-4 \le t \le 4$.

We start by constructing the following table and filling it in:

t	-4	-3	-2	-1	0	1	2	3	4
$x = t^2$	16	9	4	1	0	1	4	9	16
$y = \frac{t^3}{5}$	-12.8	-5.4	-1.6	-0.2	0	0.2	1.6	5.4	12.8

Note that you can use as many or as little t values in your table as you like. In this case using the 9 integers in the range $-4 \le t \le 4$ is enough for us to figure out the shape. You can always use more values if you feel the need.

Plotting our points then sketching the curve that goes through all of them:



Points of intersection

Further problems will involve the use of coordinate geometry. You will often need to find intersections between curves defined parametrically and functions given in cartesian form.

With such questions, the general procedure is to substitute your parametric equations into your cartesian equation, resulting in an equation for t which should be solved. The solutions to this equation represent the values of t where the two functions intersect.

Example 5: Find the points of intersection of the parabola $x = t^2, y = 2t$ with the circle
$x^2 + y^2 - 9x + 4 = 0.$

Substituting $x = t^2, y = 2t$ into the circle:	$(t^2)^2 + (2t)^2 - 9(t^2) + 4 = 0$ $\Rightarrow t^4 + 4t^2 - 9t^2 + 4 = 0$ $\Rightarrow t^4 - 5t^2 + 4 = 0$
Solving the quadratic:	The solutions to this equation via the quadratic formula are $t = -1, 1, 2, \text{ or } -2$
To find the points, we need to substitute these values of t back into the given parameterisation $x = t^2, y = 2t$. Doing so, starting with $t = -1$ and 1 :	$x = 1 \quad x = 1$ $y = -2 \quad y = 2$
Now with $t = 2$ and -2	$x = 4 \quad x = 4$ $y = 4 \quad y = -4$
Writing our solutions as coordinates:	$\therefore$ our points are (1,2) , (1,-2), (4,-4) and (4,4)

Modelling with parametric equations

You need to be able to use your knowledge of parametric equations to solve problems involving real-life scenarios. The mathematical techniques used for such problems are no different to regular questions, but in order to succeed you need to make sure you fully understand the scenario given in the question, so take some time to read through the question properly.

Mechanics problems are a popular choice for modelling questions.

Example 6: The path of a skateboarder from the point of leaving a ramp to the point of landing is modelled using the parametric equations

$x = 25t, \quad y = -4.9t^2 + 4t + 15, \quad 0 \le t \le k$

where x is the horizontal distance in meters from the point of leaving the ramp and y is the height in metres above ground level of the skateboarder, after t seconds.

a) Find the initial height of the skateboarder.
b) Find the value of k and hence state the time taken for the skateboarder to complete his jump.
c) Find the horizontal distance the skateboarder jumps.
d) Show that the skateboarder's path is a parabola according to the given model and find the maximum height above ground level of the skateboarder.

a) The initial height is the value of y at $t = 0$.	$\Rightarrow y = 15$
b) k is the value of t when the skateboarder finally lands. This is when $y = 0$. Solving $y = 0$:	$-4.9t + 4t^2 + 15 = 0 \Rightarrow t = 2.205, t = -1.388$ But since t represents time, it cannot be negative. So time taken $= 2.21\ s$ to 3 significant figures.
c) As x represents horizontal distance, we simply need to find x at $t = k = 2.21$	$x = 25(2.205\ldots) = 55.1$ to 3 significant figures.
d) We need to convert into the cartesian form. Finding t in terms of x:	$x = 25t \therefore t = \frac{x}{25}$
Substituting into y:	So $y = -4.9\left(\frac{x}{25}\right)^2 + 4\left(\frac{x}{25}\right) + 15$ $\Rightarrow y = -\frac{49}{6250}x^2 + \frac{4}{25}x + 15$
This is the equation of a parabola, which shows that the skateboarder's path is a parabola (according to the given model). The maximum height of the skateboarder will be the maximum value of y. Differentiating y and equating to 0:	$\frac{dy}{dx} = -\frac{98}{6250}x + \frac{4}{25} = 0$
Solving for x: $y = \frac{6250}{98} \times \frac{4}{25}$:	$\Rightarrow y = \frac{500}{49}$

What Is A Parametric Equation?

In this section we explore what parametric equations are and how we can relate them to equations we are familiar with.

Parametric Equation

A parametric equation is that includes an introduce an extra, independent variable called a parameter. Usually t or θ.

Cartesian Equation

A Cartesian equation is an equation that only includes x and y.

Remember

The parametric equation for the unit circle centred at the origin:

$$x = \cos(t), \quad y = \sin(t)$$

Example

Show that $x = \cos(t), \quad y = \sin(t)$ forms a unit circle at the origin (hint: put the equation in its Cartesian form and note that the Cartesian equation of a circle is given by $(x-a)^2 + (y-b)^2 = r^2$ where (a, b) is the centre and r is the radius.

Solution:

First we square both parametric equations obtaining:

$$x^2 = \cos^2(t), \quad y = \sin^2(t)$$

We then add them obtaining:

$$x^2 + y^2 = \cos^2(t) + \sin^2(t)$$
$$x^2 + y^2 = 1$$

Which is the Cartesian equation of the unit circle centred at the origin.

Remember

General parametric equation for a circle with radius r and centre (a, b) :

$$x = r\cos(t) + a \qquad y = r\sin(t) + b$$

Example

Show that $x = r\cos(t) + a, \;\; y = r\sin(t) + b$ forms a circle centred at the origin with radius r and centre (a, b).

Remember

The Cartesian equation for a circle is given by:

$$(x-a)^2 + (y-b)^2 = r^2.$$

Solution:

First we subtract a and b from both equations respectively:

$$x - a = r\cos(t), \;\; y - b = \sin(t)$$

We then square both sides of them obtaining:

$$(x-a)^2 = r^2\cos^2(t), \;\; (y-b)^2 = r^2\sin^2(t)$$

We then add both equations together:

$$\begin{aligned}(x-a)^2 + (y-b)^2 &= r^2\cos^2(t) + r^2\sin^2(t)\\(x-a)^2 + (y-b)^2 &= r^2\big(\cos^2(t) + \sin^2(t)\big)\\(x-a)^2 + (y-b)^2 &= r^2\end{aligned}$$

Example

Write the following parametric equations $x = t^2 - 1$ and $y = t - 3$ in Cartesian form.

Solution:

$$\begin{aligned} y &= t - 3 \\ &\implies t = y + 3 \\ &\implies x = (y+3)^2 - 1 \end{aligned}$$

Example

A curve C is given parametrically by the equations $x = t^2$ and $y = 3t$. The line $x + 2y + 9 = 0$ meets C at the point P. Find the coordinates of P.

Solution:

We know that$x = t^2$ and $y = 3t$, substituting this into the equation for our line gives:

$$\begin{aligned} t^2 + 6t + 9 &= 0 \\ (t+3)^2 &= 0 \\ &\implies t = -3 \end{aligned}$$

Substituting into our equations for x and y gives:

$$x = 9 \text{ and } y = -9$$

So P = (9,-9).

Example

A curve has parametric equations $x = \cos(2t)$ and $y = \cos^2(t)$. Find the Cartesian form of the curve.

Solution:

$$\begin{aligned} x &= \cos(2t) \\ x &= \cos^2(t) - 1 \\ \cos^2(t) &= x + 1 \\ \cos^2(t) &= y \end{aligned}$$

Hence,

$$x + 1 = y$$

A good rule of thumb is to rearrange to make t or some form of t, $\cos^2(t)$ for example, the subject of both parametric equations. Then you can equate these 2 equations in order to eliminate t. Also bear in mind trigonometric identities when solving these questions.

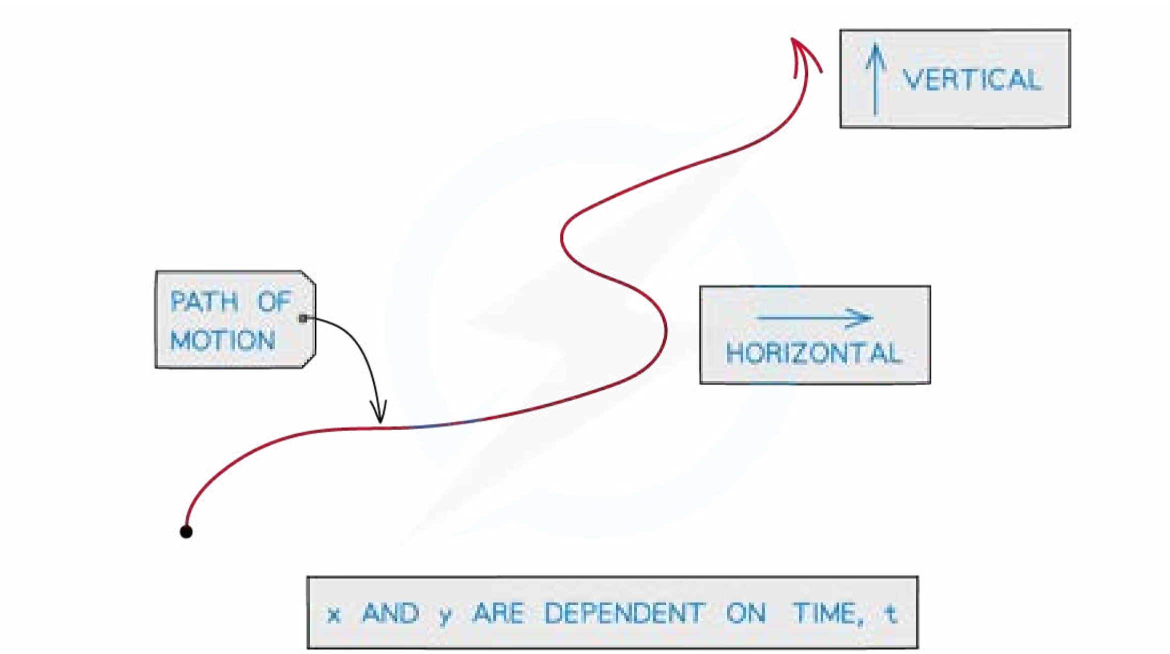
9.1.1 Parametric Equations – Basics

Your notes

Parametric Equations – Basics

What are parametric equations?

- Graphs are usually described by a **Cartesian equation**
  - The equation involves **x** and **y** only
- Equations like this can sometimes be rearranged into the form, **y = f(x)**



- In **parametric equations** both **x** and **y** are dependent on a third variable
  - This is called a **parameter**
  - **t** and **θ** are often used as parameters
- A common example ...
  - **x** is the horizontal position of an object
  - **y** is the vertical position of an object
  - and the position of the object is dependent on time **t**
- **x** is a function of **t**, **y** is a function of **t**
  - **x = f(t)**
  - **y = g(t)**

Your notes

What do I do with parametric equations?

- It is still possible to plot a graph of **y** against **x** from their parametric equations

Your notes

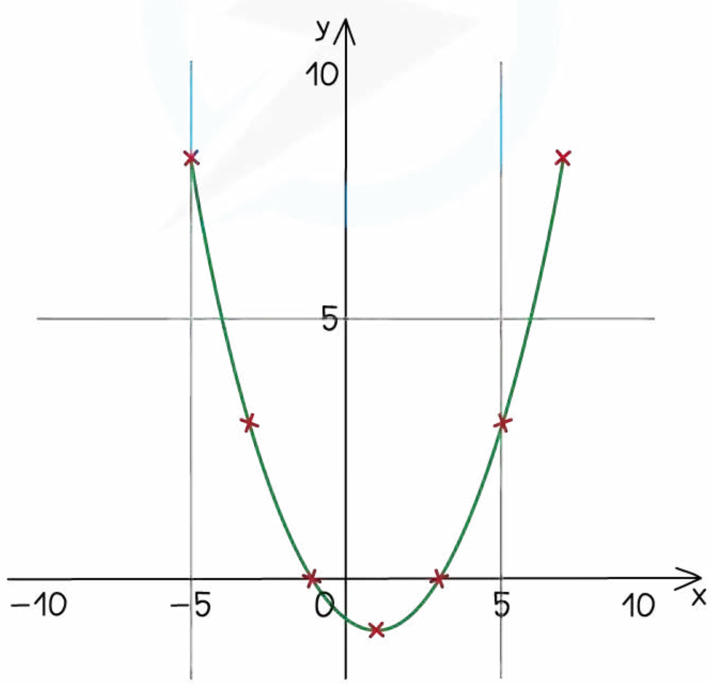
PLOTTING A GRAPH FROM PARAMETRIC EQUATIONS

e.g. PLOT THE GRAPH GIVEN BY THE PARAMETRIC EQUATIONS $x = 2t + 1$ AND $y = t^2 - 1$ FOR $-3 \leqslant t \leqslant 3$.

CONSTRUCT A TABLE OF VALUES

t	−3	−2	−1	0	1	2	3
x	−5	−3	−1	1	3	5	7
y	8	3	0	−1	0	3	8

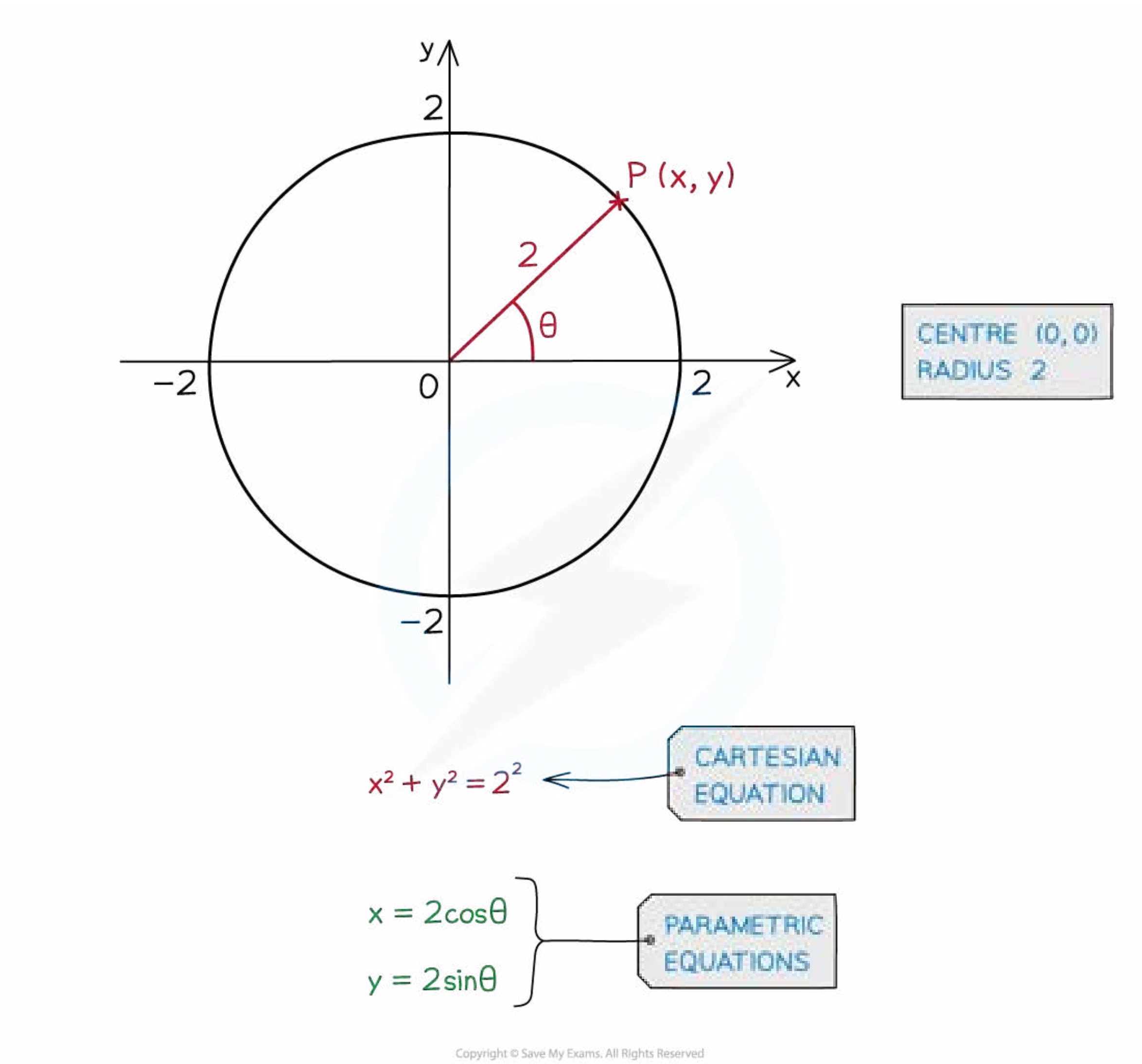
PLOT THE GRAPH OF y AGAINST x



Your notes

- Also see Parametric Equations – Sketching Graphs

What is the link between parametric equations and circles?



- For a circle, centre (0, 0) and radius **r**
  - $x = r\cos\theta$
  - $y = r\sin\theta$
  - (Note that **r** is constant, this is not two parameters)
- For a circle, centre (**a**, **b**) and radius **r**
  - $x = r\cos\theta + a$
  - $y = r\sin\theta + b$

Your notes

Worked example

(a) Write down the radius and the centre of the circle defined by the parametric equations

$$x = 3\cos\theta$$

and

$$y = 3\sin\theta - 4$$

(b) Hence write down the Cartesian equation of the circle

a) CENTRE: $(0, -4)$
RADIUS: 3

$x = r\cos\theta + a$
$y = r\sin\theta + b$

b) $x^2 + (y+4)^2 = 9$

$r^2 = 3^2 = 9$

CENTRE $(0, -4)$

9.1.2 Parametric Equations – Eliminating the Parameter

Your notes

Parametric Equations – Eliminating the Parameter

What does eliminating the parameter mean?

CARTESIAN EQUATIONS

$$y = 6x - x^2 - 5$$

$$y = 2\ln x$$

$$x^2 + y^2 = 1$$

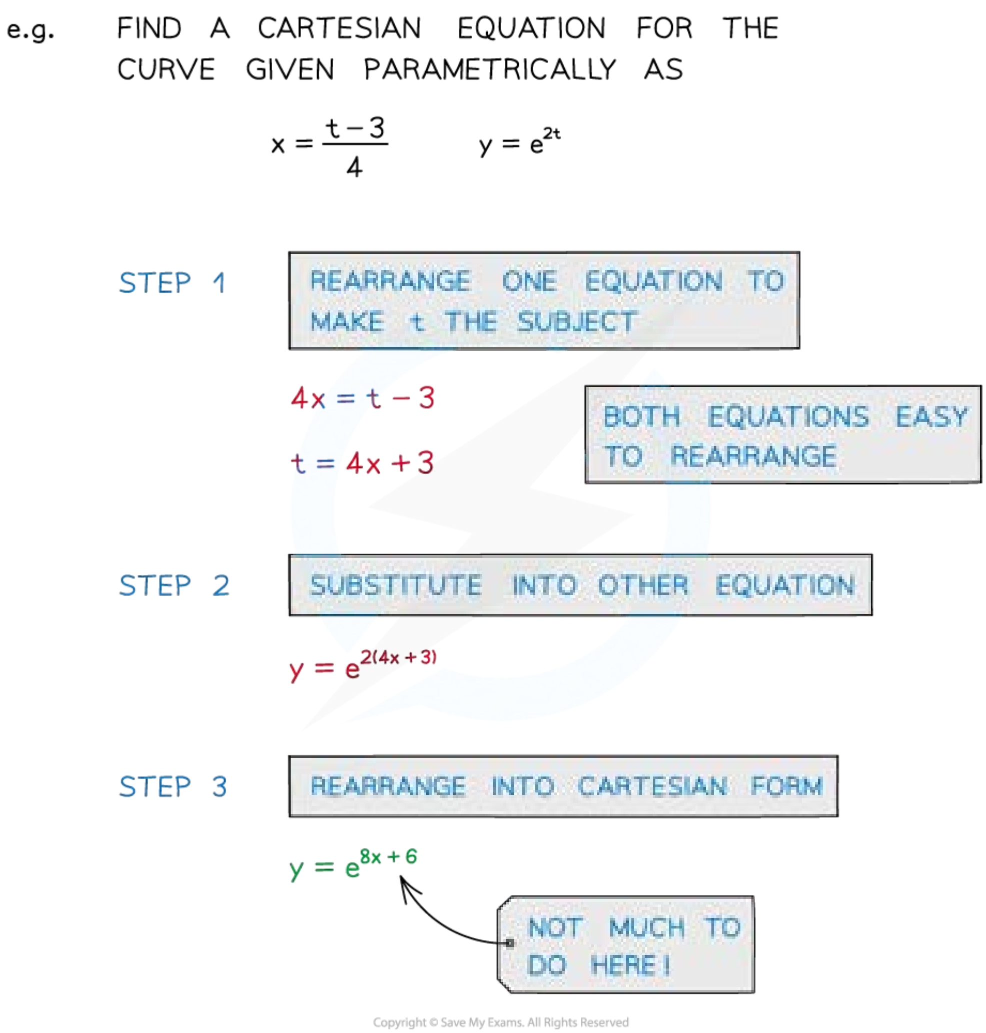
PARAMETRIC EQUATIONS

$x = t + 3$	$y = 4 - t^2$
$x = e^{2t}$	$y = 4t$
$x = \cos t$	$y = \sin t$

- In parametric equations, **x = f(t)** and **y = g(t)**
- There is still a connection directly linking **x** and **y**
  - This will be the **Cartesian** equation of the graph

How do I find the Cartesian equation from parametric equations?

Your notes



- STEP 1: Rearrange one of the equations to make t the subject
  - Either **t = p(x)** or **t = q(y)**
- STEP 2: Substitute into the other equation
- STEP 3 Rearrange into the desired (Cartesian) form

How do I eliminate t when trig is involved?

Your notes

e.g. FIND A CARTESIAN EQUATION FOR THE CURVE GIVEN PARAMETRICALLY AS

$$x = 3 + \cos t \qquad y = 2 - \sin t$$

STEP 1 — REARRANGE BOTH EQUATIONS INTO THE FORM $\cos t = \ldots$ AND $\sin t = \ldots$

$\cos t = x - 3$

$\sin t = 2 - y$

STEP 2 — SQUARE BOTH SIDES OF BOTH EQUATIONS

$\cos^2 t = (x-3)^2$

$\sin^2 t = (2-y)^2$

STEP 3 — ADD THE EQUATIONS TOGETHER

$\cos^2 t + \sin^2 t = (x-3)^2 + (2-y)^2$

STEP 4 — USE THE TRIG IDENTITY "$\sin^2 x + \cos^2 x \equiv 1$" TO ELIMINATE t

$1 = (x-3)^2 + (2-y)^2$

STEP 5 — REARRANGED INTO DESIRED FORM

$(x-3)^2 + (y-2)^2 = 1$

EQUATION OF A CIRCLE CENTRE: (3, 2)

$(y-2)^2 = (2-y)^2$

- STEP 1 Rearrange both equations into the forms "cos t = …" and "sin t = …"
- STEP 2 Square BOTH sides of BOTH equations
- STEP 3 Add the equations together
- STEP 4 The trig identity "$\sin^2 x + \cos^2 x \equiv 1$" eliminates **t**
- STEP 5 Rearrange into desired (Cartesian) form
  - This technique is seen in Trigonometric Identities

Exam Tip

When choosing which equation to rearrange, aim for "as simple as possible":

- Linear equations are simpler than quadratics
  - eg Rearrange $x = 2t + 3$
    or

    $y = 3t^2 + 3t - 4$?
- Single exponential terms are quite easy to deal with
  - eg $x = e^t \rightarrow t = \ln x$

Trig identities may be needed and remember **squared** terms are good!

- eg $\sin^2 x + \cos^2 x \equiv 1$

Worked example

Your notes

Your notes

Find the Cartesian equation for the curve C defined by the parametric equations

$$x = 3\sin 2t$$

and

$$y = 2\cos 2t$$

- THIS IS A sin/cos BASED QUESTION
- AIM FOR "SQUARE AND ADD"

STEP 1 REARRANGE BOTH INTO FORM "$\sin t = \ldots$" AND "$\cos t = \ldots$"

$$\frac{x}{3} = \sin 2t \qquad \frac{y}{2} = \cos 2t$$

STEP 2 & 3 "SQUARE AND ADD"

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = \sin^2 2t + \cos^2 2t$$

STEP 4 ELIMINATE t

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

STEP 5 REARRANGE INTO DESIRED FORM

Your notes

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

EQUATION OF AN ELIPSE

$$4x^2 + 9y^2 = 36$$

9.1.3 Parametric Equations – Sketching Graphs

Your notes

Parametric Equations – Sketching Graphs

How do I sketch a graph from parametric equations?

- **Plotting** a graph is covered in Parametric Equations - Basics

Your notes

e.g. SKETCH THE GRAPH OF y AGAINST x FOR THE PARAMETRIC EQUATIONS

$$x = 3t + 1 \qquad y = t^2 - 4$$

$3t + 1 = 0$

$t = -\frac{1}{3} \longrightarrow y = (-\frac{1}{3})^2 - 4$

$(0, -\frac{35}{9}) \qquad y = -\frac{35}{9}$

y-AXIS INTERCEPT
$x = 0$

$t^2 - 4 = 0$

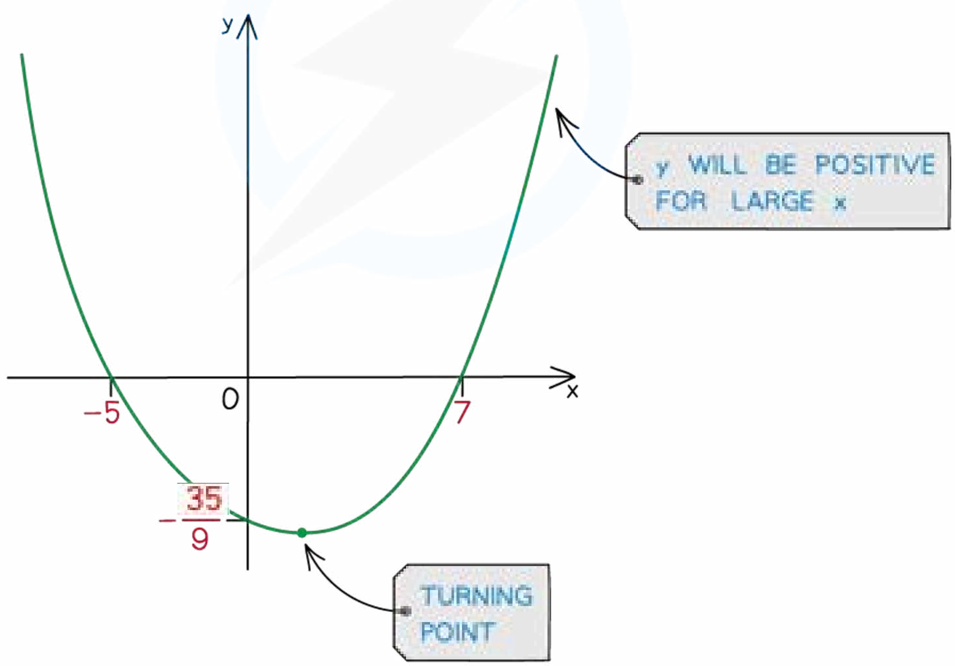
$t = \pm 2 \longrightarrow x = 3 \times 2 + 1 = 7$

$x = 3 \times -2 + 1 = -5$

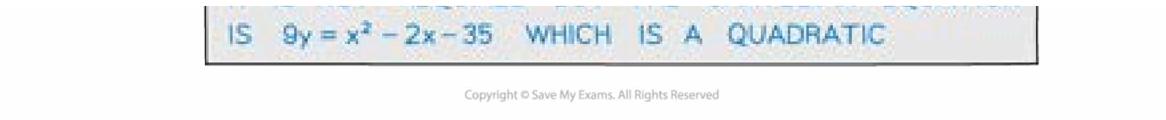
$(-5, 0)$

$(7, 0)$

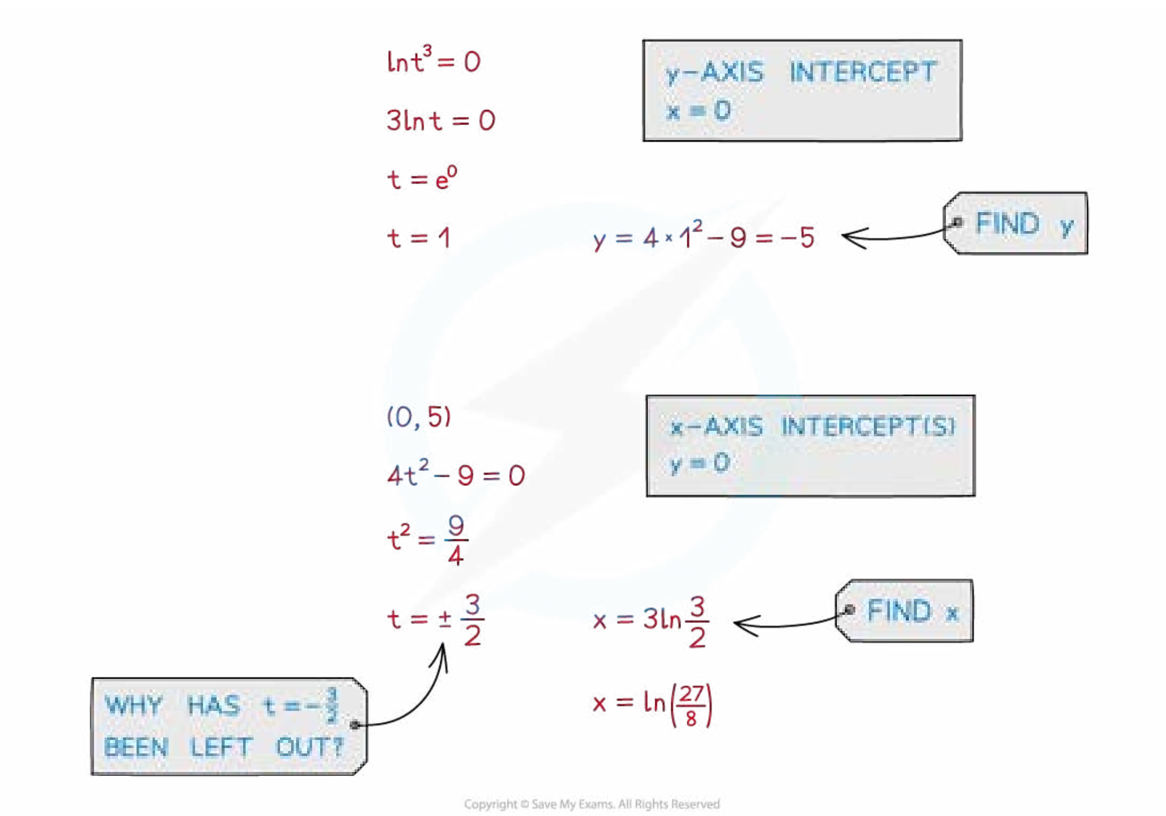
x-AXIS INTERCEPT(S)
$y = 0$



IT IS NOT REQUIRED BUT THE CARTESIAN EQUATION

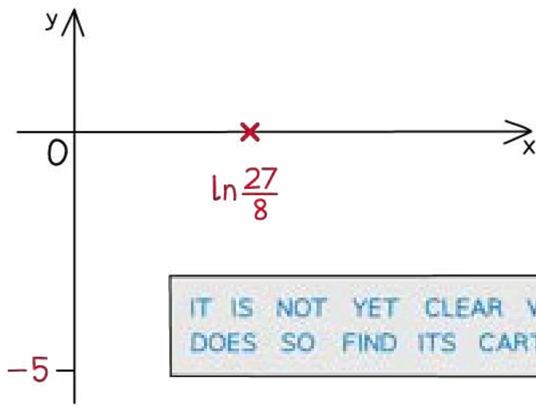


- Still find the key features of a graph ...
  - ... the **y**-axis intercept
  - ... the **x**-axis intercept(s)
  - ... asymptotes
  - ... location of (and if required coordinates of) stationary points (see Parametric Differentiation)
- **Sketch** these points and join up accordingly



Your notes

Your notes



IT IS NOT YET CLEAR WHAT ELSE THE GRAPH DOES SO FIND ITS CARTESIAN EQUATION

$$x = \ln t^3$$

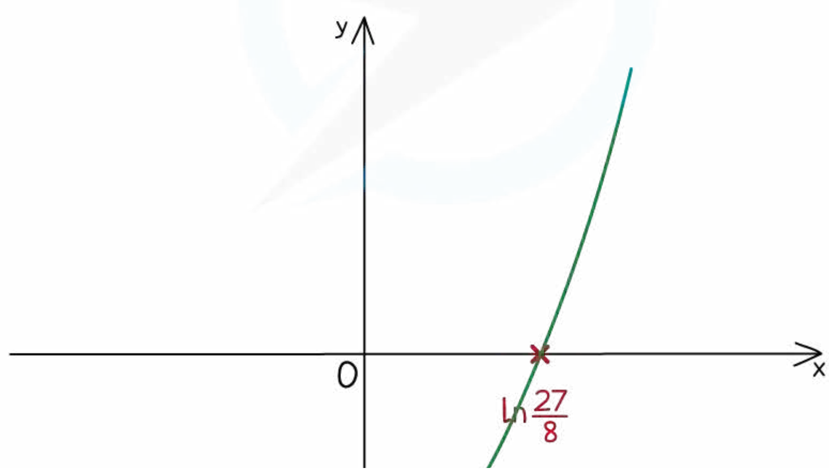
$$x = 3\ln t$$

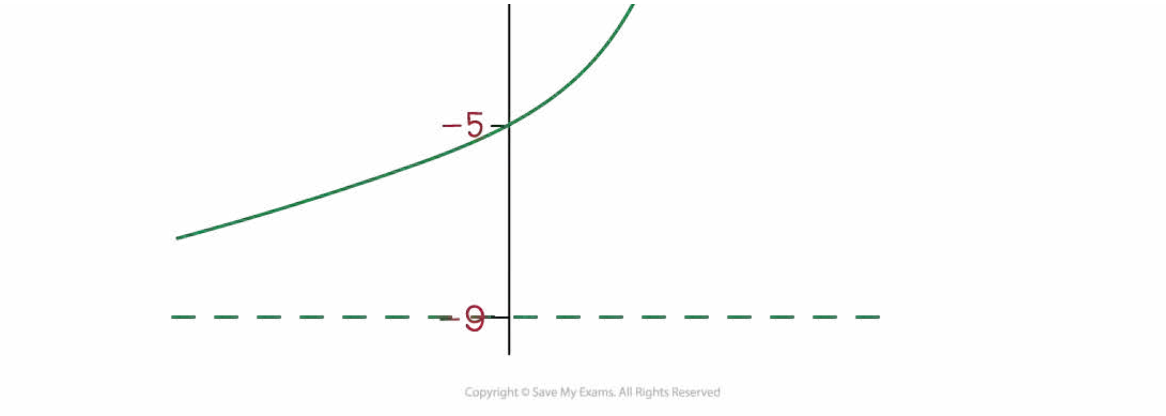
$$\ln t = \frac{x}{3}$$

$$t = e^{\frac{x}{3}} \qquad y = 4\left(e^{\frac{x}{3}}\right)^2 - 9$$

$$y = 4e^{\frac{2}{3}x} - 9$$

A POSITIVE EXPONENTIAL GRAPH
• ASYMPTOTE AT $x = -9$





Your notes

Exam Tip

- Not all curves defined parametrically lead to familiar shaped graphs and it may be worth plotting a few extra points by calculating them.
- Your calculator may be able to produce a table of values quickly, however, remember you are sketching and not plotting.
- It may be easier to find the Cartesian equation first and draw the graph from that – this will depend on the question.
- It is only a **definite** strategy if you cannot make progress otherwise.
- If you are given the sketch of a graph it is usually only for reference and can be used to check answers for axes intercepts, etc.

Worked example

Your notes

Your notes

Sketch the graph of the curve defined by the parametric equations

$$x = e^t \text{ and } y = t^2 - t$$

stating clearly the coordinates of any points where the curve intercepts the coordinate axes.

DEDUCE $x > 0$ SO y-AXIS IS AN ASYMPTOTE

$x = e^t = 0$
NO SOLUTIONS

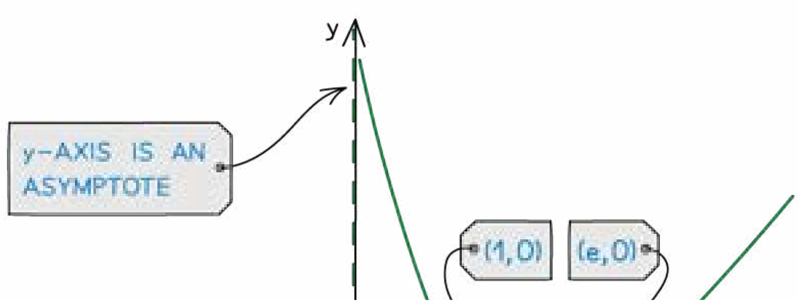
y-AXIS INTERCEPT
$x = 0$

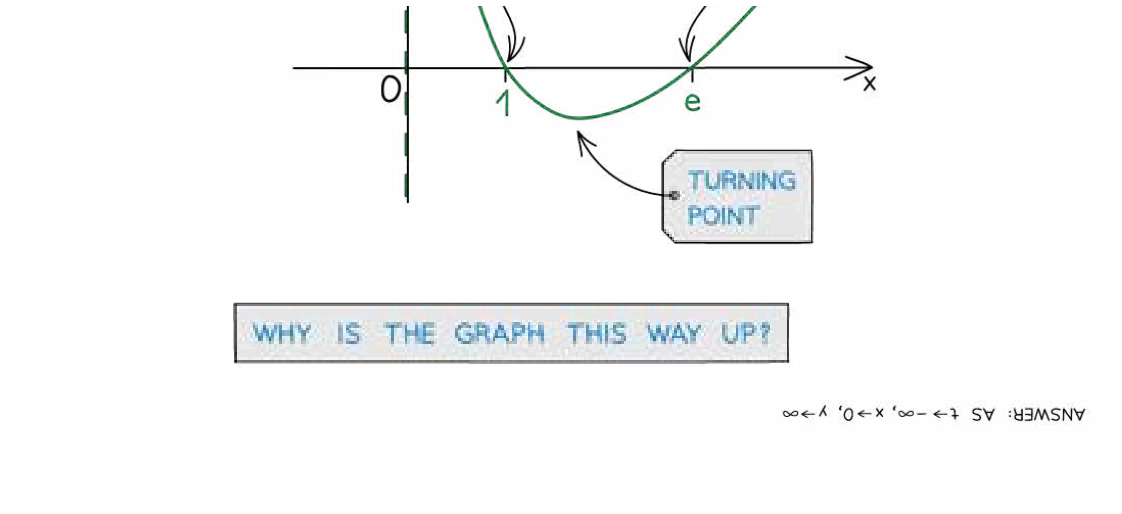
$y = t^2 - t = 0$
$t(t-1) = 0$
$t = 0, \quad t = 1$

x-AXIS INTERCEPT(S)
$y = 0$

AT $t = 0, \quad x = e^0 = 1$
AT $t = 1, \quad x = e^1 = e$

THERE WILL BE A TURNING POINT BETWEEN $x = 1$ AND $x = e$




0
1
e
x
TURNING POINT
WHY IS THE GRAPH THIS WAY UP?
ANSWER: AS t→−∞, x→0, y→∞

1. The cartesian equation of the circle C is

$$x^2 + y^2 - 8x - 6y + 16 = 0.$$

(a) Find the coordinates of the centre of C and the radius of C.

(4)

(b) Sketch C.

(2)

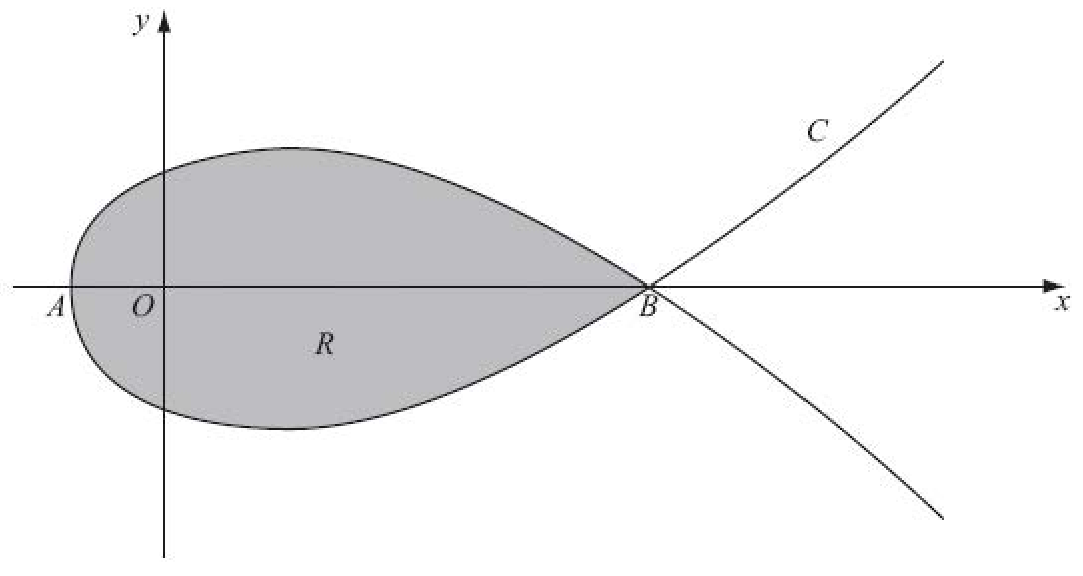
(c) Find parametric equations for C.

(3)

(d) Find, in cartesian form, an equation for each tangent to C which passes through the origin O.

(5)
(Total 14 marks)

2.



The diagram above shows a sketch of the curve C with parametric equations

$$x = 5t^2 - 4, \quad y = t(9 - t^2)$$

The curve C cuts the x-axis at the points A and B.

(a) Find the x-coordinate at the point A and the x-coordinate at the point B.

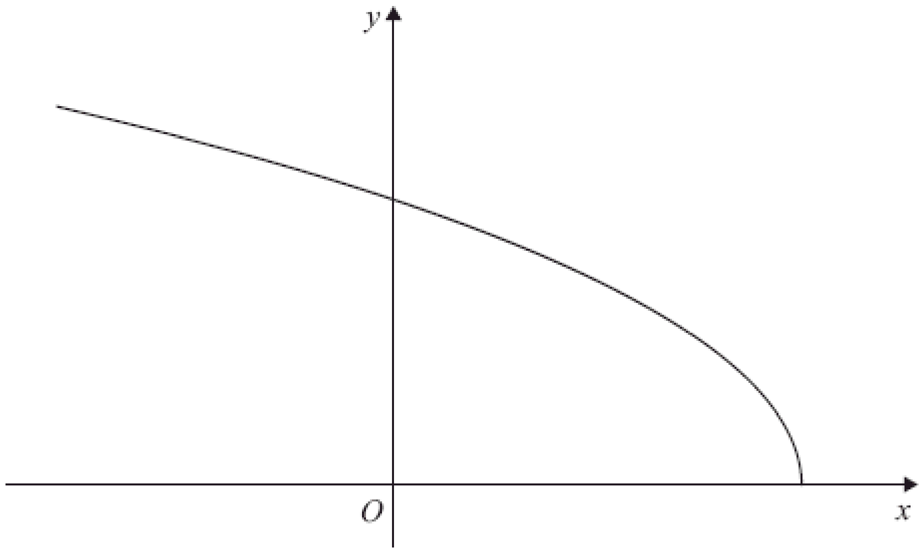
(3)

The region R, as shown shaded in the diagram above, is enclosed by the loop of the curve.

(b) Use integration to find the area of R.

(6)
(Total 9 marks)

3.



The diagram above shows a sketch of the curve with parametric equations

$$x = 2\cos 2t, \qquad y = 6\sin t, \qquad 0 \le t \le \frac{\pi}{2}$$

(a) Find the gradient of the curve at the point where $t = \frac{\pi}{3}$.

(4)

(b) Find a cartesian equation of the curve in the form

$$y = \mathrm{f}(x),\ -k \le x \le k,$$

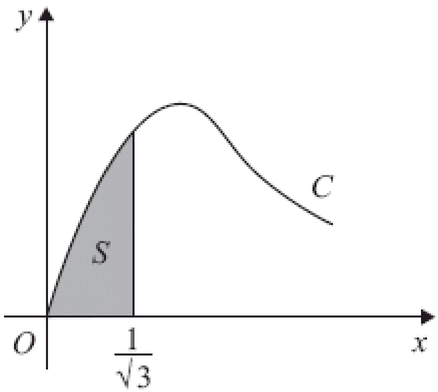
stating the value of the constant k.

(4)

(c) Write down the range of f (x).

(2)
(Total 10 marks)

4. (a) Using the identity $\cos 2\theta = 1 - 2\sin^2\theta$, find $\int \sin^2\theta \, d\theta$.



(2)

The diagram above shows part of the curve C with parametric equations

$$x = \tan\theta, \qquad y = 2\sin 2\theta, \qquad 0 \le \theta < \frac{\pi}{2}$$

The finite shaded region S shown in the diagram is bounded by C, the line $x = \frac{1}{\sqrt{3}}$ and the x-axis. This shaded region is rotated through 2π radians about the x-axis to form a solid of revolution.

(b) Show that the volume of the solid of revolution formed is given by the integral

$$k\int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta$$

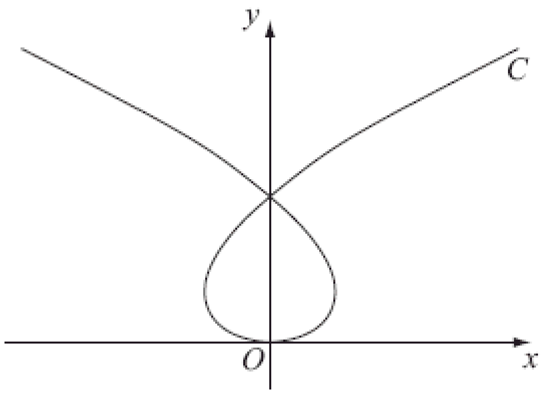
where k is a constant.

(5)

(c) Hence find the exact value for this volume, giving your answer in the form $p\pi^2 + q\pi\sqrt{3}$, where p and q are constants.

(3)
(Total 10 marks)

5.



The curve C shown above has parametric equations

$$x = t^3 - 8t,\ y = t^2$$

where t is a parameter. Given that the point A has parameter $t = -1$,

(a) find the coordinates of A.

(1)

The line l is the tangent to C at A.

(b) Show that an equation for l is $2x - 5y - 9 = 0$.

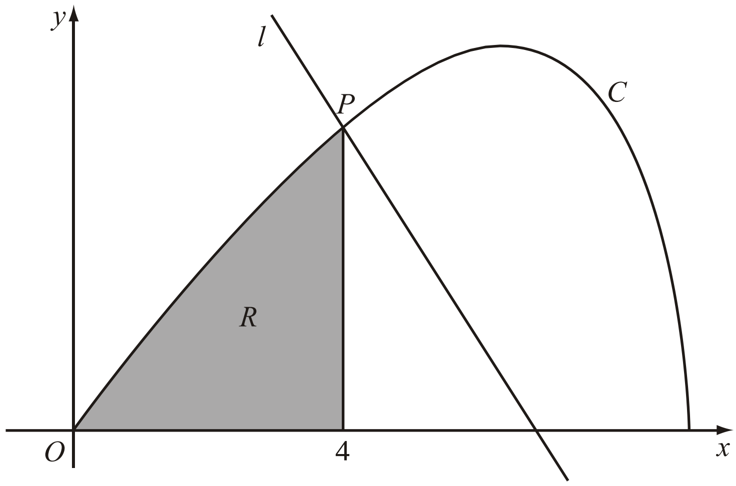
(5)

The line l also intersects the curve at the point B.

(c) Find the coordinates of B.

(6)
(Total 12 marks)

6.



The diagram above shows the curve C with parametric equations

$$x = 8\cos t, \quad y = 4\sin 2t, \quad 0 \leq t \leq \frac{\pi}{2}.$$

The point P lies on C and has coordinates $(4, 2\sqrt{3})$.

(a) Find the value of t at the point P.

(2)

The line l is a normal to C at P.

(b) Show that an equation for l is $y = -x\sqrt{3} + 6\sqrt{3}$.

(6)

The finite region R is enclosed by the curve C, the x-axis and the line $x = 4$, as shown shaded in the diagram above.

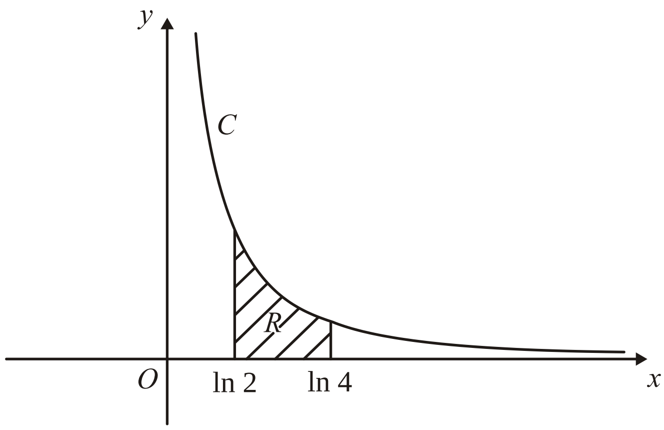
(c) Show that the area of R is given by the integral $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 64\sin^2 t\cos t\,\mathrm{d}t$.

(4)

(d) Use this integral to find the area of R, giving your answer in the form $a + b\sqrt{3}$, where a and b are constants to be determined.

(4)
(Total 16 marks)

7.



The curve C has parametric equations

$$x = \ln(t+2), \quad y = \frac{1}{(t+1)}, \quad t > -1$$

The finite region R between the curve C and the x-axis, bounded by the lines with equations x = 1n 2 and x = 1n 4, is shown shaded in the diagram above.

(a) Show that the area of R is given by the integral

$$\int_0^2 \frac{1}{(t+1)(t+2)}\,\mathrm{d}t\,.$$

(4)

(b) Hence find an exact value for this area.

(6)

(c) Find a cartesian equation of the curve C, in the form y = f(x).

(4)

(d) State the domain of values for x for this curve.

(1)
(Total 15 marks)

8. A curve has parametric equations

$$x = \tan^2 t, \qquad y = \sin t, \qquad 0 < t < \frac{\pi}{2}.$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t. You need not simplify your answer.

(3)

(b) Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{4}$.

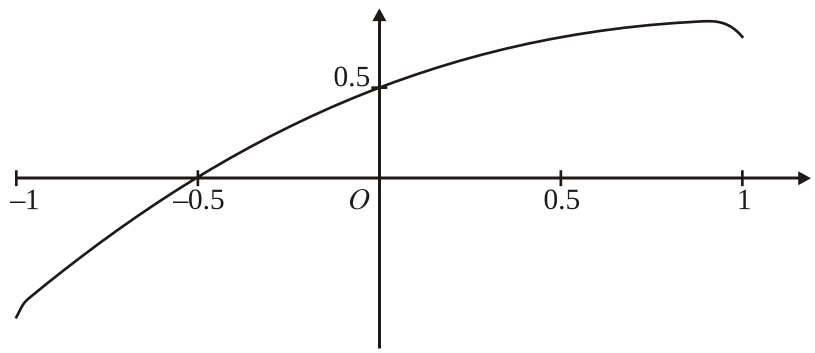
Give your answer in the form $y = ax + b$, where a and b are constants to be determined.

(5)

(c) Find a cartesian equation of the curve in the form $y^2 = f(x)$.

(4)
(Total 12 marks)

9.



The curve shown in the figure above has parametric equations

$$x = \sin t, \quad y = \sin\left(t + \tfrac{\pi}{6}\right), \qquad -\tfrac{\pi}{2} < t < \tfrac{\pi}{2}.$$

(a) Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{6}$.

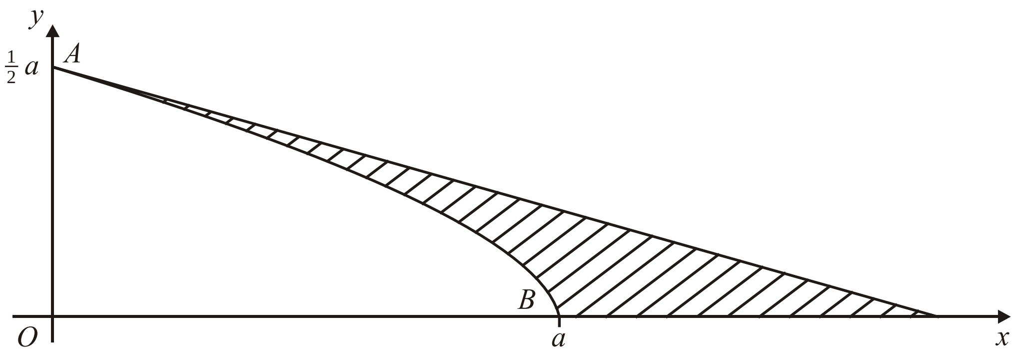
(6)

(b) Show that a cartesian equation of the curve is

$$y = \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{(1 - x^2)}, \quad -1 < x < 1$$

(3)
(Total 9 marks)

10.



The curve shown in the figure above has parametric equations

$$x = a\cos 3t,\ \ y = a\sin t, \quad 0 \le t \le \frac{\pi}{6}.$$

The curve meets the axes at points A and B as shown.

The straight line shown is part of the tangent to the curve at the point A.

Find, in terms of a,

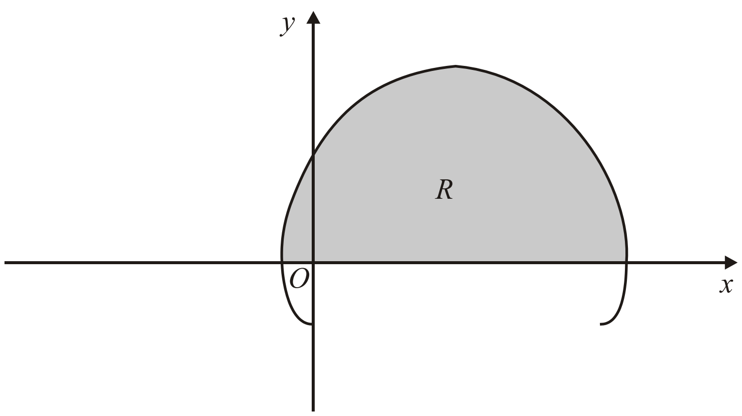
(a) an equation of the tangent at A,

(6)

(b) an exact value for the area of the finite region between the curve, the tangent at A and the x-axis, shown shaded in the figure above.

(9)
(Total 15 marks)

11.



The curve shown in the figure above has parametric equations

$$x = t - 2\sin t, \qquad y = 1 - 2\cos t, \qquad 0 \le t \le 2\pi$$

(a) Show that the curve crosses the x-axis where $t = \dfrac{\pi}{3}$ and $t = \dfrac{5\pi}{3}$.

(2)

The finite region R is enclosed by the curve and the x-axis, as shown shaded in the figure above.

(b) Show that the area of R is given by the integral

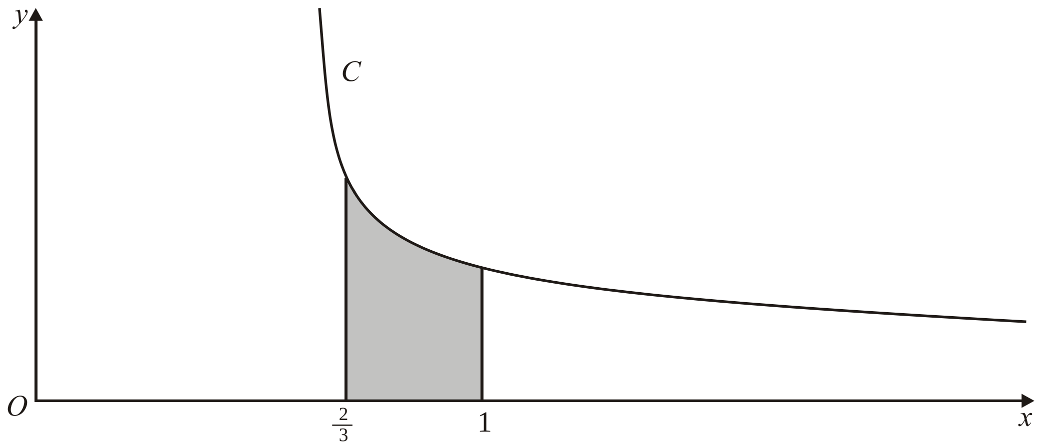
$$\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 \, \mathrm{d}t.$$

(3)

(c) Use this integral to find the exact value of the shaded area.

(7)
(Total 12 marks)

12.



The curve C has parametric equations

$$x = \frac{1}{1+t}, \quad y = \frac{1}{1-t}, \quad |t| < 1.$$

(a) Find an equation for the tangent to C at the point where $t = \tfrac{1}{2}$.

(7)

(b) Show that C satisfies the cartesian equation $y = \dfrac{x}{2x - 1}$.

(3)

The finite region between the curve C and the x-axis, bounded by the lines with equations $x = \frac{2}{3}$ and $x = 1$, is shown shaded in the figure above.

(c) Calculate the exact value of the area of this region, giving your answer in the form $a + b \ln c$, where a, b and c are constants.

(6)
(Total 16 marks)

13. A curve has parametric equations

$$x = 2 \cot t, \;\; y = 2 \sin^2 t, \;\; 0 < t \leq \frac{\pi}{2}.$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of the parameter t.

(4)

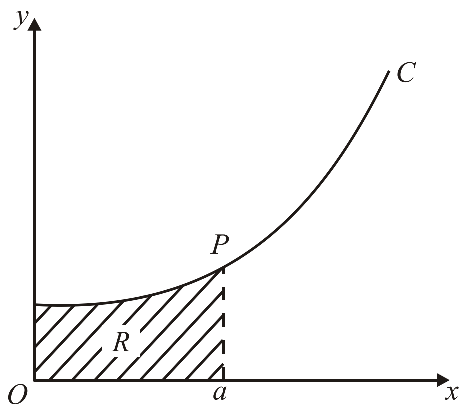
(b) Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{4}$.

(4)

(c) Find a cartesian equation of the curve in the form $y = f(x)$. State the domain on which the curve is defined.

(4)
(Total 12 marks)

14.



The diagram above shows a sketch of the curve C with parametric equations

$$x = 3t \sin t,\ y = 2 \sec t, \qquad 0 \le t < \frac{\pi}{2}.$$

The point $P(a, 4)$ lies on C.

(a) Find the exact value of a.

(3)

The region R is enclosed by C, the axes and the line $x = a$ as shown in the diagram above.

(b) Show that the area of R is given by

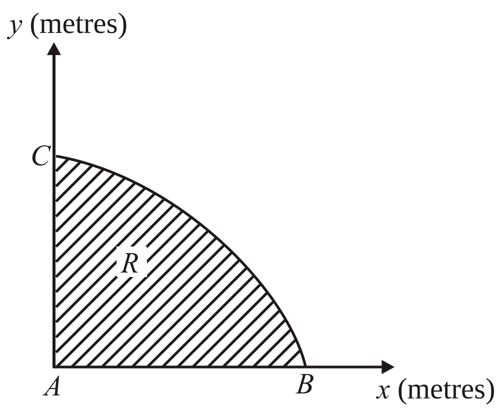
$$6\int_0^{\frac{\pi}{3}} (\tan t + t)\, dt.$$

(4)

(c) Find the exact value of the area of R.

(4)
(Total 11 marks)

15.



The diagram above shows a cross-section R of a dam. The line AC is the vertical face of the dam, AB is the horizontal base and the curve BC is the profile. Taking x and y to be the horizontal and vertical axes, then A, B and C have coordinates $(0, 0)$, $(3\pi^2, 0)$ and $(0, 30)$ respectively. The area of the cross-section is to be calculated.

Initially the profile BC is approximated by a straight line.

(a) Find an estimate for the area of the cross-section R using this approximation.

(1)

The profile BC is actually described by the parametric equations.

$$x = 16t^2 - \pi^2, \quad y = 30 \sin 2t, \quad \frac{\pi}{4} \leq t \leq \frac{\pi}{2}.$$

(b) Find the exact area of the cross-section R.

(7)

(c) Calculate the percentage error in the estimate of the area of the cross-section R that you found in part (a).

(2)
(Total 10 marks)

16. The curve C is described by the parametric equations

$$x = 3 \cos t, \quad y = \cos 2t, \quad 0 \leq t \leq \pi.$$

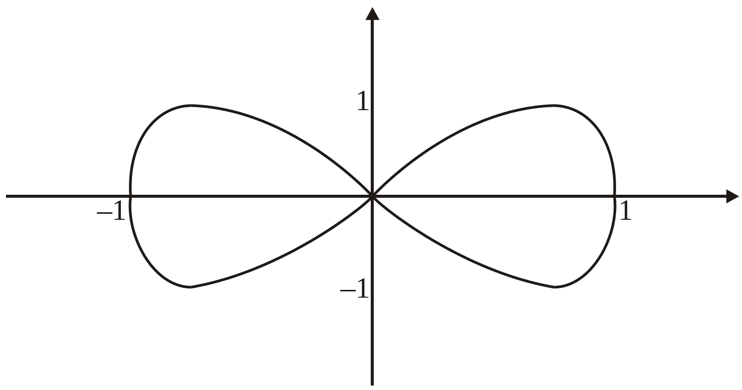
(a) Find a cartesian equation of the curve C.

(2)

(b) Draw a sketch of the curve C.

(2)
(Total 4 marks)

17.



The curve shown in the diagram above has parametric equations

$$x = \cos t, \quad y = \sin 2t, \qquad 0 \leq t < 2\pi.$$

(a) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of the parameter t.

(3)

(b) Find the values of the parameter t at the points where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

(3)

(c) Hence give the exact values of the coordinates of the points on the curve where the tangents are parallel to the x-axis.

(2)

(d) Show that a cartesian equation for the part of the curve where $0 \leq t < \pi$ is

$$y = 2x\sqrt{(1 - x^2)}.$$

(3)

(e) Write down a cartesian equation for the part of the curve where $\pi \leq t < 2\pi$.

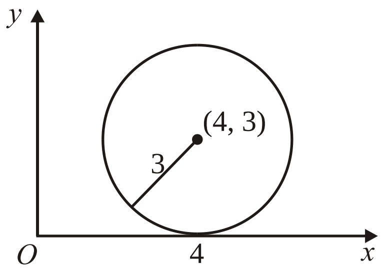
(1)
(Total 12 marks)

1. (a) $x^2 + y^2 - 8x - 6y + 16 = (x - 4)^2 - 16 + (y - 3)^2 - 9 + 16$

$(x - 4)^2 - (y - 3)^2 = 9$ M1 A1

Centre (4, 3), radius 3 A1 A1 4

(b)

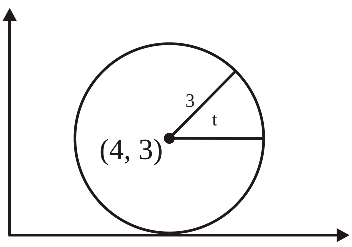


B1, B1 2

(c) $x = 4 + 3 \cos t$

$y = 3 + 3 \sin t$

$(0 \le t < 2\pi)$ M1 A1 A1 3



(d) Line through origin $y = mx$

x-coordinate of points where this line cuts C satisfies

$(1 + m^2)x^2 - 8x - 6mx + 16 = 0$ M1 A1

As line is tangent this equation has repeated roots

$(8 + 6m)^2 = 4(1 + m^2)16$

$16 + 9m^2 + 24m = 16 + 16m^2$

$24m = 7m^2$ M1 A1

$m = 0, \ m = \dfrac{24}{7}$

Equations of tangents are $y = 0$, $y = \dfrac{24}{7}x$ A1 5

[14]

2.	(a)	$y = 0 \Rightarrow t(9 - t^2) = t(3 - t)(3 + t) = 0$			
		$t = 0, 3, –3$	Any one correct value	B1	
		At $t = 0$, $x = 5(0)^2 – 4 = –4$	Method for finding one value of x	M1	
		At $t = 3$, $x = 5(3)^2 – 4 = 41$			
		$(\text{At } t = –3, x = 5(-3)^2 - 4 = 41)$			
		At A, $x = –4$; at B, $x = 41$	Both	A1	3
	(b)	$\dfrac{dx}{dt} = 10t$	Seen or implied	B1	
		$\int y\,dx = \int y\frac{dx}{dt}\,dt = \int t(9 - t^2)10t\,dt$		M1 A1	
		$= \int (90t^2 - 10t^2)\,dt$			
		$\left[\frac{90t^3}{3} - \frac{10t^5}{5}\right]_0^3 = 30 \times 3^3 - 2 \times 3^5 (= 324)$		M1	
		$A = 2\int y\,dx = 648$ (units^2)		A1	6

[9]

3.	(a)	$\dfrac{dx}{dt} = –4\sin 2t,\ \dfrac{dy}{dt}\ 6\cos t$		B1, B1	
		$\dfrac{dy}{dx} = -\dfrac{6\cos t}{4\sin 2t}\left(= -\dfrac{3}{4\sin t}\right)$		M1	
		At t = $\dfrac{\pi}{3}$, $m = -\dfrac{3}{4 \times \frac{\sqrt{3}}{2}} = -\dfrac{\sqrt{3}}{2}$	accept equivalents, awrt –0.87	A1	4

Alternatives to (a) where the parameter is eliminated

1	$y = (18 - 9x)^{\frac{1}{2}}$		
	$\dfrac{dy}{dx} = \dfrac{1}{2}(18 - 9x)^{-\frac{1}{2}} \times (–9)$	B1	
	At $t = \dfrac{\pi}{3}, x = \cos\dfrac{2\pi}{3} = -1$	B1	
	$\dfrac{dy}{dx} = \dfrac{1}{2} \times \dfrac{1}{\sqrt{(27)}} \times -9 = -\dfrac{\sqrt{3}}{2}$	M1 A1	4

2 $y^2 = 18 - 9x$

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = -9$ B1

At $t = \frac{\pi}{3}, y = 6\sin\frac{\pi}{3} = 3\sqrt{3}$ B1

$\frac{dy}{dx} = -\frac{9}{2\times 3\sqrt{3}} = -\frac{\sqrt{3}}{2}$ M1 A1 4

(b) Use of $\cos 2t = 1 - 2\sin^2 t$ M1

$\cos 2t = \frac{x}{2}, \sin t = \frac{y}{6}$

$\frac{x}{2} = 1 - 2\left(\frac{y}{6}\right)^2$ M1

Leading to $y = \sqrt{(18-9x)}\,(= 3\sqrt{(2-x)})$ cao A1

$-2 \le x \le 2$ $k = 2$ B1 4

(c) $0 \le \mathrm{f}(x) \le 6$ either $0 \le \mathrm{f}(x)$ or $\mathrm{f}(x) \le 6$ B1

Fully correct. Accept $0 \le y \le 6$, [0, 6] B1 2

[10]

4. (a) $\int \sin^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int(1-\cos 2\theta)\,\mathrm{d}\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\,(+C)$ M1 A1 2

(b) $x = \tan\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$

$\pi\int y^2\,\mathrm{d}x = \pi\int y^2\frac{\mathrm{d}x}{\mathrm{d}\theta}\mathrm{d}\theta = \pi\int(2\sin 2\theta)^2\sec^2\theta\,\mathrm{d}\theta$ M1 A1

$= \pi\int\frac{(2\times 2\sin\theta\cos\theta)^2}{\cos^2\theta}\mathrm{d}\theta$ M1

$16\pi\int\sin^2\theta\,\mathrm{d}\theta$ $k = 16\pi$ A1

$x = 0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0,\ x = \frac{1}{\sqrt{3}} \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$ B1 5

$\left(V = 16\pi\int_0^{\frac{\pi}{6}}\sin^2\theta\,\mathrm{d}\theta\right)$

	(c)	$V = 16\pi\left[\frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{6}}$		M1	
		$= 16\pi\left[\left(\frac{\pi}{12} - \frac{1}{4}\sin\frac{\pi}{3}\right) - (0-0)\right]$	Use of correct limits	*M1	
		$= 16\pi\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right) = \frac{4}{3}\pi^2 - 2\pi\sqrt{3}$	$p = \frac{4}{3}, q = -2$	A1	3

[10]

5.	(a)	At A, $x = -1 + 8 = 7$ & $y = (-1)^2 = 1 \Rightarrow A(7,1)$	$A(7, 1)$	B1	1
	(b)	$x = t^3 - 8t$, $y = t^2$,			
		$\frac{dx}{dt} = 3t^2 - 8, \frac{dy}{dt} = 2t$			
		$\therefore \frac{dy}{dx} = \frac{2t}{3t^2 - 8}$	Their $\frac{dy}{dt}$ divided by their $\frac{dx}{dt}$	M1	
			Correct $\frac{dy}{dx}$	A1	
		At A, m(**T**) = $\underline{\frac{2(-1)}{3(-1)^2 - 8}} = \underline{\frac{-2}{3-8}} = \underline{\frac{-2}{-3}} = \underline{\frac{2}{5}}$	Substitutes for t to give any of the four underlined oe:		
		T: y – (their 1) = m_r(x – (their 7)) or $1 = \frac{2}{5}(7) + c \Rightarrow c = 1 - \frac{14}{5} = -\frac{9}{5}$	Finding an equation of a tangent with their point and their tangent gradient or finds c and uses y = (their gradient)x + "c" .	dM1	
		Hence **T**: $y = \frac{2}{5}x - \frac{9}{5}$			
		gives **T**: $\underline{2x - 5y - 9 = 0}$ **AG**		A1 cso	5
	(c)	$2(t^3 - 8t) - 5t^2 - 9 = 0$	Substitution of both $x = t^3 - 8t$ and $y = t^2$ into **T**	M1	
		$2t^3 - 5t^2 - 16t - 9 = 0$			
		$(t + 1)\{(2t^2 - 7t - 9) = 0\}$			
		$(t + 1)\{(t + 1)(2t - 9) = 0\}$	A realisation that $(t + 1)$is a factor.	dM1	
		$\{t = -1(\text{at } A)\ t = \frac{9}{2} \text{ at } B\}$	$t = \frac{9}{2}$	A1	

$x = \left(\frac{9}{2}\right)^2 - 8\left(\frac{9}{2}\right) = \frac{729}{8} - 36 = \frac{441}{8} = 55.125$ or awrt 55.1

Candidate uses their value of t to find either the x or y coordinate — ddM1

$y = \left(\frac{9}{2}\right)^2 = \frac{81}{4} = 20.25$ or awrt 20.3 — One of either x or y correct. — A1

Both x and y correct. — A1 — 6

Hence $B\left(\frac{441}{8}, \frac{81}{4}\right)$ — awrt

[12]

6. (a) At $P(4,\ 2\sqrt{3})$ either $\underline{4 = 8\cos t}$ or $\underline{2\sqrt{3} = 4\sin 2t}$

$\Rightarrow$ only solution is $\underline{t = \frac{\pi}{3}}$ where $0 \leqslant t \leqslant \frac{\pi}{2}$

$\underline{4 = 8\cos t}$ or $\underline{2\sqrt{3} = 4\sin 2t}$ — M1

$\underline{t = \frac{\pi}{3}}$ or $\underline{\text{awrt } 1.05}$ (radians) only stated in the range $0 \leqslant t \leqslant \frac{\pi}{2}$ — A1 — 2

(b) $x = 8\cos t,\ y = 4\sin 2t$

$\frac{dx}{dt} = -8\sin t,\ \frac{dy}{dt} = 8\cos 2t$

At P, $\frac{dy}{dx} = \underline{\frac{8\cos\left(\frac{2\pi}{3}\right)}{-8\sin\left(\frac{\pi}{3}\right)}}$

$\left\{ = \frac{8\left(-\frac{1}{2}\right)}{(-8)\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{3}} = \text{awrt } 0.58 \right\}$

Hence m(**N**) $= -\sqrt{3}$ or $\frac{-1}{\frac{1}{\sqrt{3}}}$

N: $y - 2\sqrt{3} = -\sqrt{3}(x - 4)$

N: $\underline{y = -\sqrt{3}x + 6\sqrt{3}}$ **AG**

or $2\sqrt{3} = -\sqrt{3}(4) + c \Rightarrow c = 2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$

so **N**: $\left\lfloor \underline{y = -\sqrt{3}x + 6\sqrt{3}} \right\rfloor$

Attempt to differentiate both x and y wrt t to give $\pm p$ sin t and $\pm q$ cos $2t$ respectively	M1	
Correct $\frac{dx}{dt}$ and $\frac{dy}{dt}$	A1	
Divides in correct way round and attempts to substitute their value of t (in degrees or radians) into their $\frac{dy}{dx}$ expression.	M1*	
You may need to check candidate's substitutions for M1* **Note the next two method marks are dependent on M1***		
Uses m(**N**) = $-\frac{1}{\text{their m}(\mathbf{T})}$	dM1*	
Uses $y - 2\sqrt{3}$ = (their m_N)(x – 4) or finds c using x = 4 and $y = 2\sqrt{3}$ and uses y = (their m_N)x + "c".	dM1*	
$\underline{y = -\sqrt{3}x + 6\sqrt{3}}$	A1 **cso AG**	6

(c) $A = \int_0^4 y\,dx = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} 4\sin 2t.(-8\sin t)dt$

$$A = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -32\sin 2t.\sin t\,dt = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -32(2\sin t\cos t).\sin t\,dt$$

$$A = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -64.\sin^2 t\cos t\,dt$$

$$A = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 64.\sin^2 t\cos t\,dt$$

attempt at $A = \underline{\int y\frac{dx}{dt}dt}$	M1	
correct expression (ignore limits and dt)	A1	
Seeing sin $2t$ = 2 sin t cos t anywhere in this part.	M1	
Correct proof. Appreciation of how the negative sign affects the limits. **Note that the answer is given in the question.**	A1 **AG**	4

(d) {Using substitution $u = \sin t \Rightarrow \dfrac{du}{dt} = \cos t$}

{change limits:

when $t = \dfrac{\pi}{3}$, $u = \dfrac{\sqrt{3}}{2}$ & when $t = \dfrac{\pi}{2}$, $u = 1$}

$$A = 64\left[\frac{\sin^3 t}{3}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \text{ or } A = 64\left[\frac{u^3}{3}\right]_{\frac{\sqrt{3}}{2}}^{1}$$

$$A = 64\left[\frac{1}{3} - \left(\frac{1}{3}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right)\right]$$

$$A = 64\left(\frac{1}{3} - \frac{1}{8}\sqrt{3}\right) = \underline{\frac{64}{3} - 8\sqrt{3}}$$

(Note that $a = \dfrac{64}{3}$, $b = -8$)

$k\sin^3 t$ or ku^3 with $u = \sin t$	M1	
Correct integration ignoring limits.	A1	
Substitutes limits of either ($t = \dfrac{\pi}{2}$ and $t = \dfrac{\pi}{3}$) or ($u = 1$ and $u = \dfrac{\sqrt{3}}{2}$) and subtracts the correct way round.	dM1	
$\underline{\dfrac{64}{3} - 8\sqrt{3}}$	A1 aef **isw**	4

Aef in the form $a + b\sqrt{3}$, with awrt 21.3 and anything that cancels to $a = \dfrac{64}{3}$ and $b = -8$.

[16]

7 (a) $\left[x = \ln(t+2),\ y = \dfrac{1}{t+1}\right], \Rightarrow \dfrac{dx}{dt} = \dfrac{1}{t+2}$

Area (R) = $\displaystyle\int_{\ln 2}^{\ln 4} \frac{1}{t+1}\,dx; = \int_0^2 \left(\frac{1}{t+1}\right)\left(\frac{1}{t+2}\right)dt$

Changing limits, when:

$x = \ln 2 \Rightarrow \ln 2 = \ln(t+2) \Rightarrow 2 = t + 2 \Rightarrow t = 0$

$x = \ln 4 \Rightarrow \ln 4 = \ln(t+2) \Rightarrow 4 = t + 2 \Rightarrow t = 2$

Hence, Area (R) = $\displaystyle\int_0^2 \frac{1}{(t+1)(t+2)}\,dt$

Must state $\frac{dx}{dt} = \frac{1}{t+2}$ B1

Area = $\int \frac{1}{t+1} dx$. Ignore limits. M1;

$\int \left(\frac{1}{t+1}\right) \times \left(\frac{1}{t+2}\right) dt$. Ignore limits. A1 **AG**

changes limits $x \to t$ so that ln2 $\to$ 0 and ln4 $\to$ 2 B1 4

(b) $\left(\frac{1}{(t+1)(t+2)}\right) = \frac{A}{(t+1)} + \frac{B}{(t+2)}$

1 = $A(t + 2) + B(t + 1)$

Let t = –1, 1 = A(1) $\Rightarrow$ <u>A = 1</u>

Let t = –2, 1 = B(–1) $\Rightarrow$ <u>B = –1</u>

$\int_0^2 \frac{1}{(t+1)(t+2)} dt = \int_0^2 \frac{1}{(t+1)} - \frac{1}{(t+2)} dt$

$= [\ln(t+1) - \ln(t+2)]_0^2$

= (ln 3 – ln 4) – (ln1 – ln 2)

Takes out brackets: = ln 3 – ln 4 + ln 2 = ln 3 – ln 2 = $\ln\left(\frac{3}{2}\right)$

$\frac{A}{(t+1)} + \frac{B}{(t+2)}$ with A and B found M1

Finds both A and B correctly.
Can be implied. A1

Writing down $\frac{1}{(t+1)(t+2)} = \frac{1}{(t+1)} + \frac{1}{(t+2)}$ means first M1A0.

Writing down $\frac{1}{(t+1)(t+2)} = \frac{1}{(t+1)} - \frac{1}{(t+2)}$ means first M1A1.

Either ± a ln(t + 1) or ± b ln(t + 2) dM1

Both ln terms correctly ft. A1ft

Substitutes ***both*** limits of 2 and 0 and subtracts the correct way round. ddM1

<u>ln 3 – ln 4 + ln 2</u> or $\underline{\ln\left(\frac{3}{4}\right) - \ln\left(\frac{1}{2}\right)}$ or <u>ln 3 – ln 2</u> or $\underline{\ln\left(\frac{3}{2}\right)}$

(must deal with ln 1) A1 aef isw 6

(c) $x = \ln(t + 2),\ y = \dfrac{1}{t+1}$

$e^x = t + 2 \Rightarrow t = e^x - 2$

$y = \dfrac{1}{e^x - 2 + 1} \Rightarrow y = \dfrac{1}{e^x - 1}$

Attempt to make t = ... the subject	M1	
giving $t = e^x - 2$	A1	
Eliminates t by substituting in y	dM1	
giving $y = \dfrac{1}{e^x - 1}$	A1	4

Aliter
Way 2

$t + 1 = \dfrac{1}{y} \Rightarrow t = \dfrac{1}{y} - 1 \text{ or } t = \dfrac{1-y}{y}$

$y(t + 1) = 1 \Rightarrow yt + y = 1 \Rightarrow yt = 1 - y \Rightarrow t = \dfrac{1-y}{y}$

$x = \ln\left(\dfrac{1}{y} - 1 + 2\right) \text{ or } x = \ln\left(\dfrac{1-y}{y} + 2\right)$

$x = \ln\left(\dfrac{1}{y} + 1\right)$

$e^x = \dfrac{1}{y} + 1$

$e^x - 1 = \dfrac{1}{y}$

$y = \dfrac{1}{e^x - 1}$

Attempt to make t =... the subject	M1	
Giving either $t = \dfrac{1}{y} - 1$ or $t = \dfrac{1-y}{y}$	A1	
Eliminates t by substituting in x	dM1	
giving $y = \dfrac{1}{e^x - 1}$	A1	4

Aliter
Way 3

$e^x = t + 2 \Rightarrow t + 1 = e^x - 1$

$y = \dfrac{1}{t+1} \Rightarrow y = \dfrac{1}{e^x - 1}$

Attempt to make $t + 1 = \ldots$ the subject	M1	
giving $t + 1 = e^x - 1$	A1	
Eliminates t by substituting in y	dM1	
giving $y = \dfrac{1}{e^x - 1}$	A1	4

Aliter
Way 4

$t + 1 = \dfrac{1}{y} \Rightarrow t + 2 = \dfrac{1}{y} + 1 \text{ or } t + 2 = \dfrac{1+y}{y}$

$x = \ln\left(\dfrac{1}{y} + 1\right) \text{ or } x = \ln\left(\dfrac{1+y}{y}\right)$

$x = \ln\left(\dfrac{1}{y} + 1\right)$

$e^x = \dfrac{1}{y} + 1 \Rightarrow e^x - 1 = \dfrac{1}{y}$

$y = \dfrac{1}{e^x - 1}$

Attempt to make $t + 2 = \ldots$ the subject	M1	
Either $t + 2 = \dfrac{1}{y} + 1 \text{ or } t + 2 = \dfrac{1+y}{y}$	A1	
Eliminates t by substituting in x	dM1	
giving $y = \dfrac{1}{e^x - 1}$	A1	4

(d) Domain: $\underline{x > 0}$

$\underline{x > 0}$ or just > 0	B1	1

[15]

8. (a) $x = \tan^2 t$, $y = \sin t$

$$\frac{dx}{dt} = 2(\tan t)\sec^2 t, \frac{dy}{dt} = \cos t \quad (*)$$

$$\therefore \frac{dy}{dx} = \frac{\cos t}{2\tan t \sec^2 t}\left(= \frac{\cos^4 t}{2\sin t}\right)$$ 3

B1 Correct $\frac{dx}{dt}$ and $\frac{dy}{dt}$

M1 $\dfrac{\pm\cos t}{\text{their } \frac{dx}{dt}}$

A1ft (*) $\dfrac{+\cos t}{\text{their } \frac{dx}{dt}}$

(b) When $t = \frac{\pi}{4}$, $x = 1$, $y = \frac{1}{\sqrt{2}}$ (need values)

When $t = \frac{\pi}{4}$, $\text{m}(\mathbf{T}) = \frac{dy}{dx} = \dfrac{\cos\frac{\pi}{4}}{2\tan\frac{\pi}{4}\sec^2\frac{\pi}{4}}$

$$= \frac{\frac{1}{\sqrt{2}}}{\underline{2.(1)\left(\frac{1}{\frac{1}{\sqrt{2}}}\right)^2}} = \frac{\frac{1}{\sqrt{2}}}{\underline{2.(1)\left(\frac{1}{\frac{1}{2}}\right)}} = \frac{\frac{1}{\sqrt{2}}}{\underline{2.(1)(2)}} = \frac{1}{\underline{4\sqrt{2}}} = \underline{\frac{\sqrt{2}}{8}}$$

B1, B1 The point $\underline{\left(1, \frac{1}{\sqrt{2}}\right)}$ or <u>(1, awrt 0.71)</u>

These coordinates can be implied.

($y = \sin\left(\frac{\pi}{4}\right)$ is not sufficient for B1)

B1 aef any of the five underlined expressions or awrt 0.18

T: $y - \frac{1}{\sqrt{2}} = \frac{1}{4\sqrt{2}}(x-1)$ (Note: The x and y coordinates must be the right way round.)

T: $\underline{y = \frac{1}{4\sqrt{2}}x + \frac{3}{4\sqrt{2}}}$ or $\underline{y = \frac{\sqrt{2}}{8}x + \frac{3\sqrt{2}}{8}}$ (*)

or $\frac{1}{\sqrt{2}} = \frac{1}{4\sqrt{2}}(1) + c \Rightarrow c = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}} = \frac{3}{4\sqrt{2}}$

(*) A candidate who incorrectly differentiates $\tan^2 t$ to give $\frac{dx}{dt} = 2\sec^2 t$ or $\frac{dx}{dt} = \sec^4 t$ is then able to fluke the correct answer in part (b).
Such candidates can potentially get: (a) B0M1A1ft (b) B1B1B1M1A0 **cso**.
Note: cso means "correct solution only".
Note: part (a) not fully correct implies candidate can achieve a maximum of 4 out of 5 marks in part (b).

Hence **T**: $\underline{y = \frac{1}{4\sqrt{2}}x + \frac{3}{4\sqrt{2}}}$ or $\underline{y = \frac{\sqrt{2}}{8}x + \frac{3\sqrt{2}}{8}}$ 5

M1ft aef Finding an equation of a tangent with *their point* and *their tangent gradient* or finds c using y = (their gradient)x + "c".

A1 aef cso Correct simplified EXACT equation of tangent

(c) **Way 1**

$x = \tan^2 t = \frac{\sin^2 t}{\cos^2 t}$ $\quad y = \sin t$

$x = \frac{\sin^2 t}{1-\sin^2 t}$

$x = \frac{y^2}{1-y^2}$

$x(1-y^2) = y^2 \Rightarrow x - xy^2 = y^2$

$x = y^2 + xy^2 \Rightarrow x = y^2(1+x)$

$y^2 = \frac{x}{1+x}$ 4

M1 Uses $\cos^2 t = 1 - \sin^2 t$

M1 Eliminates 't' to write an equation involving x and y.

ddM1 Rearranging and factorising with an attempt to make y^2 the subject.

A1 $\frac{x}{1+x}$

Aliter
Way 2

$1 + \cot^2 t = \operatorname{cosec}^2 t$
$= \dfrac{1}{\sin^2 t}$

Hence, $1 + \dfrac{1}{x} = \dfrac{1}{y^2}$

Hence, $y^2 = 1 - \dfrac{1}{(1+x)}$ or $\dfrac{x}{1+x}$

M1 Uses $1 + \cot^2 t = \operatorname{cosec}^2 t$

M1 implied Uses $\operatorname{cosec}^2 t = \dfrac{1}{\sin^2 t}$

ddM1 Eliminates 't' to write an equation involving x and y.

A1 $1 - \dfrac{1}{(1+x)}$ or $\dfrac{x}{1+x}$

Aliter
Way 3

$x = \tan^2 t$ $y = \sin t$
$1 + \tan^2 t = \sec^2 t$
$= \dfrac{1}{\cos^2 t}$
$= \dfrac{1}{1-\sin^2 t}$

Hence, $1 + x = \dfrac{1}{1-y^2}$

Hence, $y^2 = 1 - \dfrac{1}{(1+x)}$ or $\dfrac{x}{1+x}$

M1 Uses $1 + \tan^2 t = \sec^2 t$

M1 Uses $\sec^2 t = \dfrac{1}{\cos^2 t}$

ddM1 Eliminates 't' to write an equation involving x and y.

A1 $1 - \dfrac{1}{(1+x)}$ or $\dfrac{x}{1+x}$

Aliter
Way 4

$y^2 = \sin^2 t = 1 - \cos^2 t$

$= 1 - \dfrac{1}{\sec^2 t}$

$= 1 - \dfrac{1}{(1 + \tan^2 t)}$

Hence, $y^2 = 1 - \dfrac{1}{(1+x)}$ or $\dfrac{x}{1+x}$

M1 Uses $\sin^2 t = 1 - \cos^2 t$

M1 Uses $\cos^2 t = \dfrac{1}{\sec^2 t}$

ddM1 then uses $\sec^2 t = 1 + \tan^2 t$

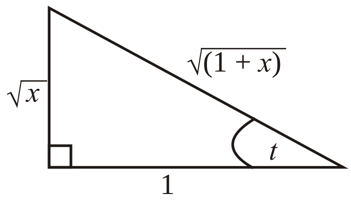
A1 $1 - \dfrac{1}{(1+x)}$ or $\dfrac{x}{1+x}$

$\dfrac{1}{1+\frac{1}{x}}$ is an acceptable response for the final accuracy A1 mark.

Aliter
Way 5

$x = \tan^2 t$ $y = \sin t$

$x = \tan^2 t \Rightarrow \tan t = \sqrt{x}$



Hence, $y = \sin t = \dfrac{\sqrt{x}}{\sqrt{1+x}}$

M1 Draws a right-angled triangle and places both $\sqrt{x}$ and 1 on the triangle

M1 Uses Pythagoras to deduce the hypotenuse

ddM1 Eliminates 't' to write an equation involving x and y

A1 $\dfrac{x}{1+x}$

$\dfrac{1}{1+\frac{1}{x}}$ is an acceptable response for the final accuracy A1 mark.

There are so many ways that a candidate can proceed with part (c).
If a candidate produces a correct solution then please award all four marks.
if they use a method commensurate with the five ways as detailed on the mark scheme then award the marks appropriately. If you are unsure of how to apply the scheme please escalate your response up to your team leader.

[12]

9. (a) x = sin t y = sin(t + $\frac{\pi}{6}$) M1

Attempt to differentiate both x and y wrt t to give two terms in cos

$\dfrac{dx}{dt} = \cos t,\ \dfrac{dy}{dt} = \cos\left(t + \frac{\pi}{6}\right)$ A1

Correct $\frac{dx}{dt}$ *and* $\frac{dy}{dt}$

When $t = \dfrac{\pi}{6}$,

$\dfrac{dy}{dx} = \underline{\dfrac{\cos\left(\frac{\pi}{6}+\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)} = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{1}{\sqrt{3}} = \text{awrt } 0.58}$ A1

Divides in correct way and substitutes for t to give any of the four underlined oe:
Ignore the double negative if candidate has differentiated $\sin \to -\cos$

when $t = \dfrac{\pi}{6}$, $x = \dfrac{1}{2}$, $y = \dfrac{\sqrt{3}}{2}$ B1

The point $\underline{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)}$ *or* $\underline{\left(\frac{1}{2}, awrt\ 0.87\right)}$

T: $\underline{y - \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}\left(x - \frac{1}{2}\right)}$

Finding an equation of a tangent with their point and their tangent gradient or finds c and uses
y = (their gradient) x + "c". dM1
Correct <u>*EXACT*</u> *equation of* <u>*tangent*</u>
oe. A1 oe

or $\frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}\left(\frac{1}{2}\right) + c \Rightarrow c = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$

or T: $\left\lfloor \underline{y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}} \right\rfloor$ 6

(b) $y = \sin\left(t + \frac{\pi}{6}\right) = \sin t \cos\frac{\pi}{6} + \cos t \sin\frac{\pi}{6}$ M1

Use of compound angle formula for sine.

Nb: $\sin^2 t + \cos^2 t = 1 \Rightarrow \cos^2 t \equiv 1 - \sin^2 t$

$\therefore\ x = \sin t$ gives $\cos t = \sqrt{(1 - x^2)}$ M1

Use of trig identity to find cos t in terms of x or $\cos^2 t$ *in terms of x.*

$\therefore\ y = \frac{\sqrt{3}}{2}\sin t + \frac{1}{2}\cos t$

gives $y = \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{(1-x^2)}$ AG A1 cso 3

Substitutes for sin *t*, cos $\frac{\pi}{6}$ *, cost and sin* $\frac{\pi}{6}$ *to give y in terms of x.*

[9]

Aliter Way 2

(a) $x = \sin t$ $y = \sin\left(t + \frac{\pi}{6}\right) = \sin t + \cos\frac{\pi}{6} + \cos t \sin\frac{\pi}{6}$ M1

(Do not give this for part (b))

Attempt to differentiate x and y wrt t to give $\frac{dx}{dy}$ *in terms of cos and* $\frac{dy}{dt}$ *in the form ± a cost ± b sin t*

$\frac{dx}{dt} = \cos t$; $\frac{dy}{dt} = \cos t \cos\frac{\pi}{6} - \sin t \sin\frac{\pi}{6}$ A1

Correct $\frac{dx}{dt}$ *and* $\frac{dy}{dt}$

When $t = \frac{\pi}{6}$, $\underline{\frac{dy}{dx} = \frac{\cos\frac{\pi}{6}\cos\frac{\pi}{6} - \sin\frac{\pi}{6}\sin\frac{\pi}{6}}{\cos\left(\frac{\pi}{6}\right)}}$ A1

Divides in correct way and substitutes for t to give any of the four underlined oe

When $t = \frac{\pi}{6}$, $x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$ B1

The point $\underline{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)}$ *or* $\underline{\left(\frac{1}{2}, \text{awrt } 0.87\right)}$

T: $\underline{y - \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}\left(x - \frac{1}{2}\right)}$

Finding an equation of a tangent with their point and their tangent gradient or finds c and uses y = (their gradient)x + "c". dM1

Correct EXACT equation of *tangent* *oe.* A1 oe

or $\frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}\left(\frac{1}{2}\right) + c \Rightarrow c = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$

or **T**: $\left\lfloor \underline{y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}} \right\rfloor$

Aliter Way 3

(a) $y = \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{(1-x^2)}$

$\frac{dy}{dx} = \frac{\sqrt{3}}{2} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(1-x^2)^{-\frac{1}{2}}(-2x)$

Attempt to differentiate two terms using the chain rule for the second term. M1

Correct $\frac{dy}{dx}$ A1

$\frac{dy}{dx} = \underline{\frac{\sqrt{3}}{2} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(1-(0.5)^2\right)^{-\frac{1}{2}}(-2(0.5)) = \frac{1}{\sqrt{3}}}$ A1

Correct substitution of x = $\frac{1}{2}$ *into a correct* $\frac{dy}{dx}$

When $t = \frac{\pi}{6}$, $x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$ B1

The point $\underline{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)}$ *or* $\underline{(\frac{1}{2}, \textit{awrt } 0.87)}$

T: $\underline{y - \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}(x - \frac{1}{2})}$

Finding an equation of a tangent with their point and their tangent gradient or finds c and uses y = (their gradient) x + "c" dM1

Correct *__EXACT__* *equation of* *__tangent__* *oe.* A1 oe

or $\frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}\left(\frac{1}{2}\right) + c \Rightarrow c = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$

or **T**: $\left\lfloor \underline{y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}} \right\rfloor$ 6

Aliter Way 2

(b) $x = \sin t$ gives $y = \frac{\sqrt{3}}{2}\sin t + \frac{1}{2}\sqrt{(1-\sin^2 t)}$ M1

Substitutes x = $\sin t$ *into the equation give in y.*

Nb: $\sin^2 t + \cos^2 t \equiv 1 \Rightarrow \cos^2 t \equiv 1 - \sin^2 t$

Cost = $\sqrt{(1-\sin^2 t)}$ M1

Use of trig identity to deduce that cos t = $\sqrt{(1-\sin^2 t)}$

gives $y = \frac{\sqrt{3}}{2}\sin t + \frac{1}{2}\cos t$

Hence $y = \sin t \cos\frac{\pi}{6} + \cos t \sin\frac{\pi}{6} = \sin(t + \frac{\pi}{6})$ A1 cso 3

Using the compound angle formula to prove $y = \sin(t + \frac{\pi}{6})$

10. (a) $\frac{dx}{dt} = -3a\sin 3t$, $\frac{dy}{dt} = a\cos t$ therefore $\frac{dy}{dx} = \frac{\cos t}{-3\sin 3t}$ M1 A1

When $x = 0$, $t = \frac{\pi}{6}$ B1

Gradient is $-\frac{\sqrt{3}}{6}$ M1

Line equation is $(y - \frac{1}{2}a) = -\frac{\sqrt{3}}{6}(x - 0)$ M1 A1 6

(b) Area beneath curve is $\int a\sin t(-3a\sin 3t)dt$ M1

$= -\frac{3a^2}{2}\int(\cos 2t - \cos 4t)dt$ M1

$\frac{3a^2}{2}[\frac{1}{2}\sin 2t - \frac{1}{4}\sin 4t]$ M1 A1

Uses limits 0 and $\frac{\pi}{6}$ to give $\frac{3\sqrt{3}a^2}{16}$ A1

Area of triangle beneath tangent is $\frac{1}{2} \times \frac{a}{2} \times \sqrt{3}a = \frac{\sqrt{3}a^2}{4}$ M1 A1

Thus required area is $\frac{\sqrt{3}a^2}{4} - \frac{3\sqrt{3}a^2}{16} = \frac{\sqrt{3}a^2}{16}$ A1 9

N.B. The integration of the product of two sines is worth 3 marks
(lines 2 and 3 of to part (b))
If they use parts

$\int \sin t \sin 3t dt = -\cos t \sin 3t + \int 3\cos 3t \cos t dt$ M1

$= -\cos t \sin 3t + 3\cos 3t \sin t + \int 9\sin 3t \sin t dt$

$8I = \cos t \sin 3t - 3\cos 3t \sin t$ M1 A1

[15]

11. (a) Solves y = 0 $\Rightarrow$ $\cos t = \frac{1}{2}$ to obtain $t = \frac{\pi}{3}$ *or* $\frac{5\pi}{3}$

(need both for A1) M1 A1 2

Or substitutes **both** values of t and shows that $y = 0$

(b) $\frac{dx}{dt} = 1 - 2\cos t$ M1 A1

Area = $\int y\,dx = \int_{\pi/3}^{5\pi/3} (1-2\cos t)\,(1-2\cos t)\,dt$

$= \int_{\pi/3}^{5\pi/3} (1-2\cos t)^2\,dt$ AG B1 3

(c) Area $= \int 1 - 4\cos t + 4\cos^2 t\,dt$ 3 terms M1

$\int 1 - 4\cos t + 2(\cos 2t + 1)dt$ (use of correct double angle formula) M1

$= \int 3 - 4\cos t + 2\cos 2t\,dt$ M1 A1

$= [3t - 4\sin t + \sin 2t]$ M1 A1

Substitutes the two correct limits $t = \frac{5\pi}{3}$ and $\frac{\pi}{3}$ and subtracts. M1

$= 4\pi + 3\sqrt{3}$ A1A1 7

[12]

12. (a) $\frac{dx}{dt} = -\frac{1}{(1+t)^2}$ and $\frac{dy}{dt} = \frac{1}{(1-t)^2}$ B1, B1

$\therefore \frac{dy}{dx} = \frac{-(1+t)^2}{(1-t)^2}$ and at $t = ½$, gradient is –9 M1 A1cao

M1 requires their dy/dt / their dx/dt and substitution of t.

At the point of contact $x = \frac{2}{3}$ and $y = 2$ B1

Equation is $y - 2 = -9\,(x - \frac{2}{3})$ M1 A1 7

(b)	**Either** obtain t in terms of x and y i,e, $t = \frac{1}{x} - 1$ or $t = 1 - \frac{1}{x}$ (or both)	M1	
	Then substitute into other expression $y = f(x)$ or $x = g(y)$ and rearrange (or put $\frac{1}{x} - 1 = 1 - \frac{1}{y}$ and rearrange)	M1	
	To obtain $y = \frac{x}{2x-1}$ (*)	a1	3
	Or Substitute into $\frac{x}{2x-1} = \frac{\frac{1}{(1+t)}}{\frac{2}{1+t} - 1}$	M1	
	$= \frac{1}{2-(1+t)} = \frac{1}{1-t}$	A1	
	$= y$ (*)	M1	3
(c)	Area = $\int_{\frac{2}{3}}^{1} \frac{x}{2x-1} \mathrm{d}x$	B1	
	$= \int \frac{u+1}{2u} \frac{\mathrm{d}u}{2} = \frac{1}{4} \int 1 + \frac{1}{u} \mathrm{d}u$ *putting into a form to integrate*	M1	
	$= \left[\frac{1}{4}u + \frac{1}{4}\ln u\right]_{\frac{1}{3}}^{1}$	M1 A1	
	$= \frac{1}{4} - \left(\frac{1}{12} + \frac{1}{4}\ln\frac{1}{3}\right)$	M1	
	$= \frac{1}{6} + \frac{1}{4}$ ln 3 or any correct equivalent.	A1	6

[16]

	Or Area = $\int_{\frac{2}{3}}^{1} \frac{x}{2x-1} \mathrm{d}x$	B1	
	$= \int \frac{1}{2} + \frac{\frac{1}{2}}{2x-1} \mathrm{d}x$ *putting into a form to integrate*	M1	
	$= \left[\frac{1}{2}x + \frac{1}{4}\ln(2x-1)\right]_{\frac{2}{3}}^{1}$	M1 A1	

$= \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\ln\frac{1}{3} = \frac{1}{6} - \frac{1}{4}\ln\frac{1}{3}$	dM1 A1	6
Or Area $= \int \frac{1}{1-t}\frac{-1}{(1+t)^2}\,\mathrm{d}t$	B1	
$= \int \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(1+t)^2}\,\mathrm{d}t$	M1	
putting into a form to integrate		
$= \left[\frac{1}{4}\ln(1-t) - \frac{1}{4}\ln(1+t) + \frac{1}{2}(1+t)^{-1}\right]$	M1 a1ft	
= Using limits 0 and ½ and subtracting (either way round)	dM1	
$= \frac{1}{6} + \frac{1}{4}$ ln3 or any correct equivalent.	A1	6
Or Area $= \int_{\frac{2}{3}}^{1} \frac{x}{2x-1}\,\mathrm{d}x$ then use parts	B1	
$= \frac{1}{2}x\ln(2x-1) - \int_{\frac{2}{3}}^{1} \frac{1}{2}(2x-1)\,\mathrm{d}x$	M1	
$= \frac{1}{2}x\ln(2x-1) - \left[\frac{1}{4}(2x-1)\ln(2x-1) - \frac{1}{2}x\right]$	M1A1	
$= \frac{1}{2} - \left(\frac{1}{3}\ln\frac{1}{3} - \frac{1}{12}\ln\frac{1}{3} + \frac{1}{3}\right)$	DM1	
$= \frac{1}{6} - \frac{1}{4}\ln\frac{1}{3}$	A1	6

13.	(a)	$\frac{\mathrm{d}x}{\mathrm{d}t} = -2\mathrm{cosec}^2\, t,\ \frac{\mathrm{d}y}{\mathrm{d}t} = 4\sin t\cos t$	both M1 A1	
		$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\sin t\cos t}{\mathrm{cosec}^2 t}$ $(= -2\sin^3 t\cos t)$	M1 A1	4
	(b)	At $t = \frac{\pi}{4}$, $x = 2$, $y = 1$	B1	
		both x and y		
		Substitutes $t = \frac{\pi}{4}$ into an attempt at $\frac{\mathrm{d}y}{\mathrm{d}x}$ to obtain gradient $\left(-\frac{1}{2}\right)$	M1	
		Equation of tangent is $y - 1 = -\frac{1}{2}(x - 2)$	M1 A1	4
		Accept $x + 2y = 4$ or any correct equivalent		

	(c)	Uses $1 + \cot^2 t = \operatorname{cosec}^2 t$, or equivalent, to eliminate t	M1	
		$1 + \left(\frac{x}{2}\right)^2 = \frac{2}{y}$	A1	
		correctly eliminates t		
		$y = \frac{8}{4 + x^2}$	cao A1	
		The domain ir $x \ldots 0$	B1	4
		Alternative for (c):		
		$\sin t = \left(\frac{y}{2}\right)^{\frac{1}{2}}$; $\cos t = \frac{x}{2}\sin t = \frac{x}{2}\left(\frac{y}{2}\right)^{\frac{1}{2}}$		
		$\sin^2 t + \cos^2 t = 1 \Rightarrow \frac{y}{2} + \frac{x^2}{4} \times \frac{y}{2} = 1$	M1 A1	
		Leading to $y = \frac{8}{4 + x^2}$	A1	

[12]

14.	(a)	$4 = 2\sec t \Rightarrow \cos t = \frac{1}{2}, \Rightarrow t = \frac{\pi}{3}$	M1, A1	
		$\therefore\ a = 3 \times \frac{\pi}{3} \times \sin\frac{\pi}{3} = \underline{\frac{\pi\sqrt{3}}{2}}$	B1	3
	(b)	$A = \int_0^a y\,dx = \int y\frac{dx}{dt}\,dt$	M1	
		Change of variable		
		$= \int 2\sec t \times [3\sin t + 3t\cos t]\,dt$	M1	
		Attempt $\frac{dx}{dt}$		
		$= \int_0^{\frac{\pi}{3}} (6\tan t, + 6t)\,dt$ (*)	A1, A1cso	4
		Final A1 requires limit <u>*stated*</u>		
	(c)	$A = [6\ln\sec t + 3t^2]_0^{\frac{\pi}{3}}$	M1, A1	
		Some integration (M1) both correct (A1) ignore lim.		
		$= (6\ln 2 + 3 \times \frac{\pi^2}{9}) - (0)$ Use of $\frac{\pi}{3}$	M1	
		$= \underline{6\ln 2 + \frac{\pi^2}{3}}$	A1	4

[11]

15. (a) Area of triangle = $\frac{1}{2} \times 30 \times 3\pi^2$ (= 444.132) B1 1

Accept 440 or 450

(b) **Either** Area shaded = $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 30 \sin 2t.32t\,dt$ M1 A1

$= [-480t \cos 2t + \int 480 \cos 2t]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ M1 A1

$= [-480t \cos 2t + 240 \sin 2t]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ A1 ft

$= 240(\pi - 1)$ M1 A1 7

or $\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 60 \cos 2t.(16t^2 - \pi^2)dt$ M1 A1

$= [(30 \sin 2t\,(\pi^2 - 16t^2) - 480t \cos 2t + \int 480 \cos 2t]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ M1 A1

$= [-480t \cos 2t + 240 \sin 2t]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ A1 ft

$= 240(\pi - 1)$ M1 A1 7

(c) Percentage error = $\frac{240(\pi - 1) - \text{estimate}}{240(\pi - 1)} \times 100 = 13.6\%$ M1 A1 2

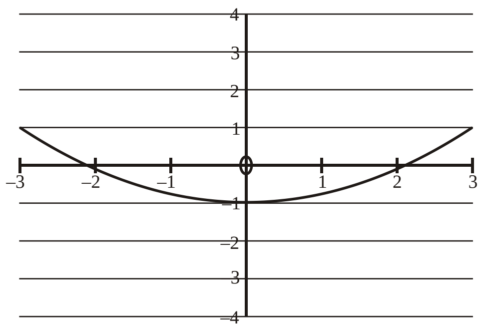
(Accept answers in the range 12.4% to 14.4%)

[10]

16. (a) Attempt to use correctly stated double angle formula $\cos 2t = 2\cos^2 t - 1$, or complete method using other double angle formula for $\cos 2t$ with $\cos^2 t + \sin^2 t = 1$ to eliminate t and obtain y = M1

$y = 2\left(\frac{x}{3}\right)^2 - 1$ or any correct equivalent.(even $y = \cos 2(\cos^{1}\left(\frac{x}{3}\right))$ A1 2

(b)



shape	B1	
position including restricted domain $-3 < x < 3$	B1	2

[4]

17. (a) $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = 2\cos 2t$ $\therefore \frac{dy}{dx} = \frac{2\cos 2t}{-\sin t}$ M1 A1 A1 3

(b) $2\cos 2t = 0$ $\therefore 2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ M1

So $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ A1 A1 3

(c) $\left(\frac{1}{\sqrt{2}}, 1\right)$ $\left(\frac{1}{\sqrt{2}}, -1\right)$ $\left(-\frac{1}{\sqrt{2}}, 1\right)$ $\left(-\frac{1}{\sqrt{2}}, -1\right)$ M1 A1 2

(d) $y = 2\sin t\cos t$ M1

$= 2\sqrt{1-\cos^2 t}\cos t = 2x\sqrt{1-x^2}$ M1 A1 3

(e) $y = -2x\sqrt{1-x^2}$ B1 1

[12]

Year 2: A Level Mathematics	
Integration	
Self-Assessment: Please identify areas in which you believe are your strong points and those you feel you need to improve on Provide evidence to support your assessment with reference to the content in this booklet.	
Strengths	**Areas for Improvement**

Name: ____ Date: ________________

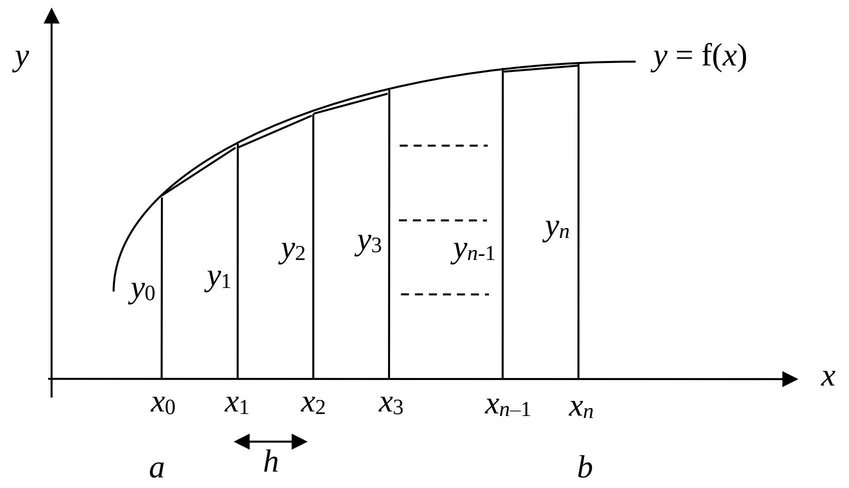
Numerical Integration: Trapezium Rule

Yr2/A level Pure Maths

Learning objectives

- Be able to apply the trapezium rule to estimate the value of an integral
- Know how to improve on an estimate by the trapezium rule

We can approximate the area beneath a curve using this method.



The area is divided into **strips**. Each strip is a **trapezium** (on its side.)
Each x-value has a corresponding y-value.
Each y-value is called an **ordinate**. If there are n strips there are $n + 1$ ordinates.
Each strip is has **width *h*.**

The area of 1st trapezium is $\frac{1}{2} \times h \times (y_0 + y_1)$

The area of 2nd trapezium is $\frac{1}{2} \times h \times (y_1 + y_2)$ etc

The area of last trapezium is $\frac{1}{2} \times h \times (y_{n-1} + y_n)$

Total = $\frac{1}{2} \times h \times (y_0 + y_1) + \frac{1}{2} \times h \times (y_1 + y_2) + \frac{1}{2} \times h \times (y_2 + y_3) + \ldots + \frac{1}{2} \times h \times (y_{n-1} + y_n)$

$$\boxed{\int_a^b f(x)\ dx \approx \frac{1}{2} h\ \{y_0 + y_n + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1})\}}$$

See formula book

Another way of saying this is:

$$\boxed{\frac{1}{2} \times \text{width of the strips} \times \{\text{first } y \text{ value} + \text{last } y \text{ value} + 2 \times (\text{sum of other } y \text{ values})\}}$$

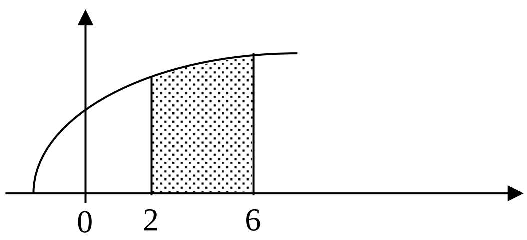
To improve the approximation, make the strips narrower (so that h is smaller) and therefore there will be more strips in the interval.

Note, also, that in the diagram above, the nature of the curve will result in the approximation always being an underestimate as all of the trapezia lie below the curve.

Example

Estimate the area beneath the curve $y = \sqrt{x+1}$ from $x = 2$ to $x = 6$ using the trapezium rule with 5 ordinates (4 strips). Give your answer to 2 decimal places.

$f(x) = \sqrt{x+1}$ so $y_0 = f(2)$, $y_1 = f(3)$, etc



x_0	x_1	x_2	x_3	x_4
2	**3**	**4**	**5**	6
y_0	y_1	y_2	y_3	y_4
1·732	**2**	**2·236**	**2·449**	**2·646**

So, clearly, $h = 1$ the width of each strip (That is, $\frac{6-2}{4} = 1$)

To get the y-values:

$y_0 = f(2) = \sqrt{2+1} = \sqrt{3} = 1{\cdot}732$
$y_1 = f(3) = \sqrt{3+1} = \sqrt{4} = 2$
$y_2 = f(4) = \sqrt{4+1} = \sqrt{5} = 2{\cdot}236$
$y_3 = f(5) = \sqrt{5+1} = \sqrt{6} = 2{\cdot}449$
$y_4 = f(6) = \sqrt{6+1} = \sqrt{7} = 2{\cdot}646$

These have all been given to 3 decimal places. This is so that the final answer can be given accurately to 2 decimal places.

(So, to approximate the integral of other functions, evaluate the function at each of the x-values)

$$\int_a^b f(x)\ dx \approx \frac{1}{2} h\,\{y_0 + y_n + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1})\}$$

$\frac{1}{2} \times$ width of the strips $\times$ {first y value + last y value + 2 $\times$ (sum of other y values) }

So: $\displaystyle\int_2^6 \sqrt{x+1}\ dx \approx \frac{1}{2} \times 1 \times \{\, 1{\cdot}732 + 2{\cdot}646 + 2 \times (2 + 2{\cdot}236 + 2{\cdot}449)\,\} = \underline{\mathbf{8{\cdot}87}}$ (to 2 dp)

Exercise

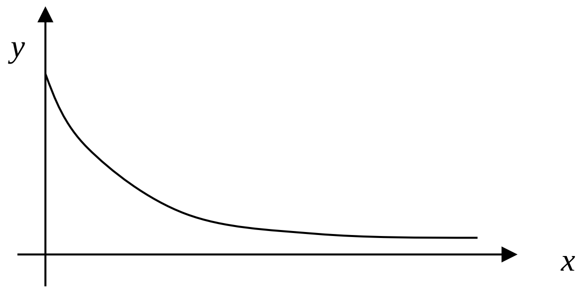
1: (a) Use the trapezium rule with five ordinates (four strips) to find an approximate value of this integral to 2 decimal places.

$$\int_0^8 \sqrt{x+2}\ \mathrm{d}x$$

x_0	x_1	x_2	x_3	x_4
0				8
y_0	y_1	y_2	y_3	y_4

(b) With the use of a diagram of the graph of $y = \sqrt{x+2}$, explain why the estimate obtained in part (a) is less than the exact value of the integral.

2: (a) The following is a sketch of the graph of $y = 4 \times 2^{-x}$



Estimate the area of the region under the curve between $x = 0$ and $x = 5$ using the trapezium rule with 6 ordinates (5 strips).

(b) How could your estimate in part (a) be improved?

3: A particle travels in a straight line with its velocity, v ms^{-1}, at time, t seconds, given by

$$v = 10 - 7e^{-2t}$$

The trapezium rule with n strips is to be used to estimate the distance travelled by the particle in its first 3 seconds of motion.

(a) Find the estimate of distance travelled when $n = 4$ (that is, 4 strips and 5 ordinates.)

(b) Sofia, a student, claims that as $n \to \infty$, the distance travelled would be

$$\frac{1}{2}(7e^{-6} + 53)$$

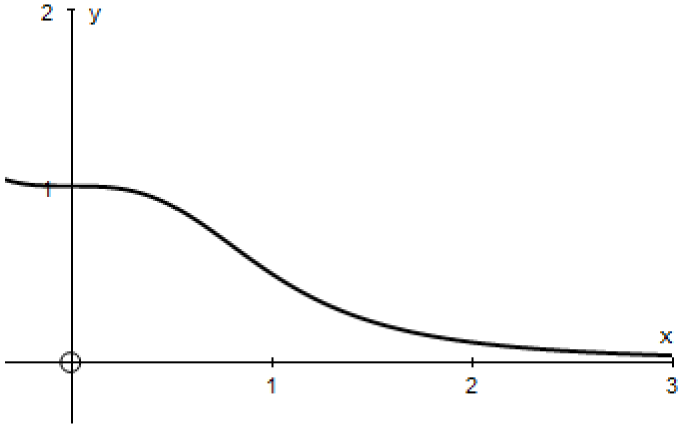
Show that Sofia is correct.

4: Use the trapezium rule with 6 ordinates (5 strips) to find an approximate value for

$$\int_{0\cdot8}^{2\cdot3} \log_{10}(x^2 + 1) \quad dx$$

giving your answer to three decimal places.

5: (a) The diagram below shows a portion of the graph of $y = \dfrac{1}{1 + x^3}$



Use the trapezium rule with five ordinates (four strips) to estimate the value of

$$\int_1^2 \frac{1}{1 + x^3}\, \mathrm{d}x$$

giving your answer to 4 decimal places.

(b) **Hence**, estimate the value of

$$\int_1^2 2 - \frac{1}{1 + x^3} \, dx$$

(c) Is the value of the estimate in part (b) an over-estimate or under-estimate of the exact value of the integral?

Numerical Integration

Yr2/A Level Further Pure

To estimate the value of $\int_a^b \mathrm{f}(x)\ \mathrm{d}x$ when f(x) is difficult to integrate.

We split the area of the region to be found into **strips**. The width of each strip is h.

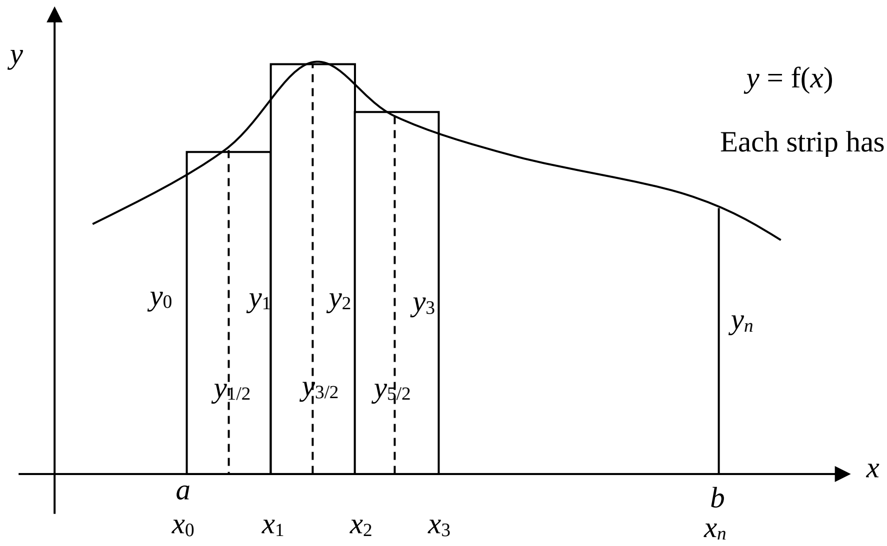
Mid-Ordinate Rule

The **middle** of each strip has a y-value the same as the height of the curve at that point (the 'ordinate'.)

The x-values are therefore called $x_{\frac{1}{2}}, x_{\frac{3}{2}}, x_{\frac{5}{2}}$ etc.

The corresponding y-values are called $y_{\frac{1}{2}}, y_{\frac{3}{2}}, y_{\frac{5}{2}}$ etc. See the diagram below.

If there are n strips then the x-values of each strip are $x_0, x_1, x_2\ldots$ etc. So $h = \dfrac{b-a}{n}$



The area of the region is estimated by adding together the area of the rectangles.
Each rectangle has an area found by **$h \times y$-value in the middle**

So $$\int_a^b \mathrm{f}(x)\ \mathrm{d}x \approx h \times y_{1/2} + h \times y_{3/2} + h \times y_{5/2} \ \ldots$$

Or $$\boxed{\int_a^b \mathrm{f}(x)\ \mathrm{d}x \approx h\,(y_{1/2} + y_{3/2} + y_{5/2}\ \ldots)}$$

Example 1

Estimate $\int_2^6 \sqrt{x+2}\ \mathrm{d}x$ using Mid-ordinates Rule with 4 strips (5 ordinates) giving your answer to 3 significant figures.

If there are 4 strips then $h = 1$ $\qquad h = \dfrac{6-2}{4} = 1$

x_0		x_1		x_2		x_3		x_4
2		3		4		5		6
	$x_{1/2}$		$x_{3/2}$		$x_{5/2}$		$x_{7/2}$	
	2·5		3·5		4·5		5·5	
	$y_{1/2}$		$y_{3/2}$		$y_{5/2}$		$y_{7/2}$	
	$\sqrt{4\cdot5}$		$\sqrt{5\cdot5}$		$\sqrt{6\cdot5}$		$\sqrt{7\cdot5}$	
	2·121		2·345		2·550		2·739	

Calculate the *y*-values by substituting 2·5, 3·5 etc into $\sqrt{x+2}$
Each is given to **four** significant figures.

$$\int_a^b f(x)\ \mathrm{d}x \approx h\,(y_{1/2} + y_{3/2} + y_{5/2} \ \dots)$$

$$\int_2^6 \sqrt{x+2}\ \mathrm{d}x \approx 1 \times (2\cdot121 + 2\cdot345 + 2\cdot550 + 2\cdot739) = \mathbf{9\cdot76} \qquad \text{(to 3 significant figures)}$$

Example 2

Estimate $\int_1^4 \ln(6x)\ \mathrm{d}x$ using the mid-ordinate rule with 6 strips (7 ordinates) giving your answer to three significant figures.

If there are 6 strips then $h = 0{\cdot}5$ $\qquad h = \dfrac{4-1}{6} = 0{\cdot}5$

x_0		x_1		x_2		x_3		x_4		x_5		x_6
1		1·5		2		2·5		3		3·5		4
	1·25		1·75		2·25		2·75		3·25		3·75	
	2·015		2·351		2·603		2·803		2·970		3·114	

Calculate the *y*-values by substituting 1·25, 1·75 etc into ln(6*x*).
Each is given to **four** significant figures.

$$\int_a^b f(x)\ \mathrm{d}x \approx h\,(y_{1/2} + y_{3/2} + y_{5/2} \ \dots)$$

$$\int_1^4 \ln(6x)\ \mathrm{d}x \approx 0\cdot5 \times (2\cdot015 + 2\cdot351 + 2\cdot603 + 2\cdot803 + 2\cdot970 + 3\cdot114) = \mathbf{7\cdot93}$$

Exercise 1: Mid-ordinate Rule

1: Use the mid-ordinate rule with 4 strips (5 ordinates) to find an estimate for $\int_{1}^{5} \sqrt{1 + x^2}\ dx$

Give your answer to 1 decimal place.

x_0		x_1		x_2		x_3		x_4
1		2		3		4		5
	$x_{1/2}$		$x_{3/2}$		$x_{5/2}$		$x_{7/2}$	
	$y_{1/2}$		$y_{3/2}$		$y_{5/2}$		$y_{7/2}$	

2: Use the mid-ordinate rule with 4 strips (5 ordinates) to find an estimate for $\int_{0}^{2} \frac{1}{x^3 + 1}\ dx$

Give your answer to 2 decimal places.

3: Use the mid-ordinate rule with 5 strips (6 ordinates) to find an estimate for $\int_0^2 \frac{x}{\sqrt{2+x^3}}\, dx$

Give your answer to 3 decimal places.

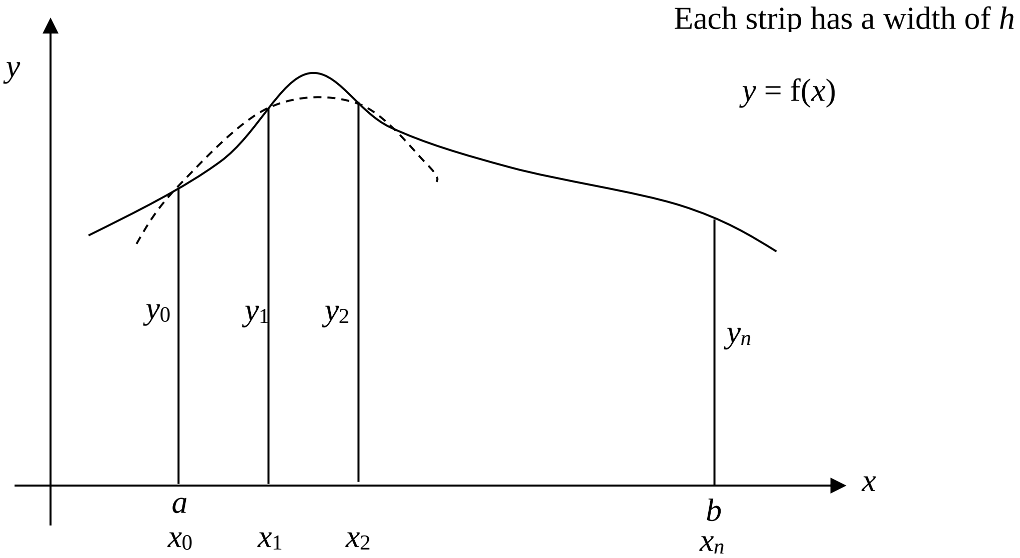
4: Use the mid-ordinate rule with 5 strips (6 ordinates) to find an estimate for $\int_1^5 \frac{1}{e^x+1}\, dx$.

Give your answer to 4 decimal places.

Simpson's Rule[1]

This is the best approximation that we study as it uses sets of three points and **fits a quadratic** to them. So **there must always be an even number of strips** (and an odd number of ordinates.)

The derivation of Simpson's rule is given in the appendix. For Core 3 you only need to be able to use the rule and you do not have to derive or even memorise it.



Area of first 2 strips $\approx \frac{1}{3}h(\, y_0 + 4y_1 + y_2\,)$

This is what Thomas Simpson worked out. See the appendix for the method.

If this is repeated for each pair of strips and added together we get:

$$\int_a^b y \quad dx \approx \frac{1}{3}h\{\,(y_0 + y_n) + 4(y_1 + y_3 + y_5 \ldots) + 2(y_2 + y_4 + y_6 \ldots)\,\}$$

$$\int_a^b y \quad dx \approx \frac{1}{3}h\,\{\,(\text{first} + \text{last}) + 4(\text{sum of } \textit{odd} \text{ ordinates}) + 2(\text{sum of } \textit{even} \text{ ordinates})\,\}$$

[1] **Thomas Simpson** FRS (20 August 1710 – 14 May 1761) was a British mathematician, inventor and eponym of Simpson's rule to approximate definite integrals. The attribution, as often in mathematics, can be debated: this rule had been found 100 years earlier by Johannes Kepler, and in German is the so-called Keplersche Fassregel.
Simpson was born in Market Bosworth, Leicestershire. The son of a weaver, Simpson taught himself mathematics, then turned to astrology after seeing a solar eclipse. He also dabbled in divination and caused fits in a girl after 'raising a devil' from her. After this incident, he and his wife had to flee to Derby. They later moved to London. From 1743, he taught mathematics at the Royal Military Academy, Woolwich.
Apparently, the method that became known as Simpson's rule was well known and used earlier by Bonaventura Cavalieri (a student of Galileo) in 1639, later rediscovered by James Gregory (who Simpson succeeded as Regius Professor of Mathematics at the University of St Andrews) and was only attributed to Simpson. Despite popular opinion, Simpson's rule was not devised by Homer Simpson, or indeed, Mrs Simpson.

Example 1

Estimate $\int_2^4 \sqrt{1+e^x} \quad dx$ using Simpson's Rule with 4 strips (5 ordinates) giving your answer to 4 decimal places.

If there are 4 strips then $h = 0{\cdot}5$ $\qquad h = \dfrac{4-2}{4} = 0{\cdot}5$

x_0	x_1	x_2	x_3	x_4
2	2·5	3	3·5	4

y_0	y_1	y_2	y_3	y_4
2·89639	3·63077	4·59190	5·84084	7·45641
First	Odd	Even	Odd	Last

Calculate the y-values by substituting 2, 2·5 etc into $\sqrt{1+e^x}$
Each is given to **five** decimal places.

$$\int_a^b y \quad \mathrm{d}x \approx \tfrac{1}{3}h\{\, (y_0 + y_n) + 4(y_1 + y_3 + y_5 \ldots) + 2(y_2 + y_4 + y_6 \ldots) \,\}$$

$$\int_a^b y \quad \mathrm{d}x \approx \tfrac{1}{3}h\ \{\, (\text{first} + \text{last}) + 4(\text{sum of } \textit{odd} \text{ ordinates}) + 2(\text{sum of } \textit{even} \text{ ordinates}) \,\}$$

$$\int_2^4 \sqrt{1+e^x} \quad dx \approx \frac{1}{3} \times 0{\cdot}5 \times \{\, (2{\cdot}89639 + 7{\cdot}45641) + 4 \times (3{\cdot}63077 + 5{\cdot}84084) + 2 \times 4{\cdot}59190 \,\}$$

$$= \underline{\mathbf{9{\cdot}5705}}$$

Example 2

Estimate $\int_0^3 4^x \quad dx$ using Simpson's Rule with 6 strips (7 ordinates.)

If there are 6 strips then $h = 0{\cdot}5$ $\qquad h = \dfrac{3-0}{6} = 0{\cdot}5$

x_0	x_1	x_2	x_3	x_4	x_5	x_6
0	0·5	1	1·5	2	2·5	3

y_0	y_1	y_2	y_3	y_4	y_5	y_6
1	2	4	8	16	32	64

$$\int_a^b y \quad \mathrm{d}x \approx \tfrac{1}{3}h\{\, (y_0 + y_n) + 4(y_1 + y_3 + y_5 \ldots) + 2(y_2 + y_4 + y_6 \ldots) \,\}$$

$$\int_a^b y \quad \mathrm{d}x \approx \tfrac{1}{3}h\ \{\, (\text{first} + \text{last}) + 4(\text{sum of } \textit{odd} \text{ ordinates}) + 2(\text{sum of } \textit{even} \text{ ordinates}) \,\}$$

$$\int_0^3 4^x \quad dx \approx \frac{1}{3} \times 0{\cdot}5 \times \{\, (1 + 64) + 4 \times (2 + 8 + 32) + 2 \times (4 + 16) \,\} = \mathbf{45{\cdot}5}$$

Exercise 2: Simpson's Rule

1: Use Simpson's rule with 4 strips (5 ordinates) to find an estimate for $\displaystyle\int_0^2 \frac{1}{e^{2x}+1}\,dx$

Give your answer to 3 decimal places.

x_0	x_1	x_2	x_3	x_4
0	0·5	1	1·5	2
y_0	y_1	y_2	y_3	y_4

2: Use Simpson's rule with 4 strips (5 ordinates) to find an estimate for $\displaystyle\int_0^8 \sqrt{x+2}\,dx$

Give your answer to 1 decimal place.

3: Use Simpson's rule with 4 strips (5 ordinates) to find an estimate for $\displaystyle\int_2^6 \frac{1}{x^3+1}\,dx$

Give your answer to 3 decimal places

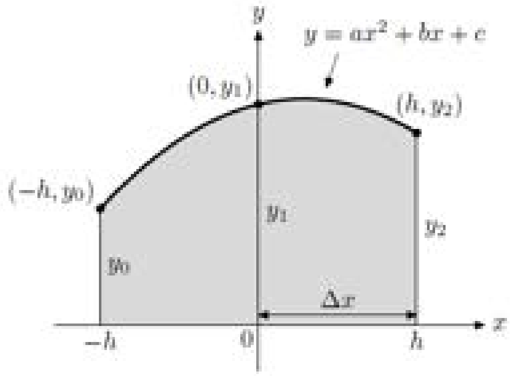
4: Use Simpson's rule with 6 strips (7 ordinates) to find an estimate for $\displaystyle\int_1^4 \frac{x^3}{x+3}\,dx$

Give your answer to 3 significant figures.

Appendix: Simpson's Rule derivation

Simpson's rule is a numerical method that approximates the value of a definite integral by using quadratic polynomials.

Let's first derive a formula for the area under a parabola of equation $y = ax^2 + bx + c$ passing through the three points: $(-h, y_0)$, $(0, y_1)$, (h, y_2).



$$
\begin{aligned}
A &= \int_{-h}^{h} (ax^2 + bx + c)\, dx \\
&= \left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx \right) \Bigg|_{-h}^{h} \\
&= \frac{2ah^3}{3} + 2ch \\
&= \frac{h}{3} \left(2ah^2 + 6c\right)
\end{aligned}
$$

Since the points $(-h, y_0)$, $(0, y_1)$, (h, y_2) are on the parabola, they satisfy $y = ax^2 + bx + c$. Therefore,

$$
\begin{aligned}
y_0 &= ah^2 - bh + c \\
y_1 &= c \\
y_2 &= ah^2 + bh + c
\end{aligned}
$$

Observe that

$$y_0 + 4y_1 + y_2 = (ah^2 - bh + c) + 4c + (ah^2 + bh + c) = 2ah^2 + 6c.$$

Therefore, the area under the parabola is

$$A = \frac{h}{3} (y_0 + 4y_1 + y_2) = \frac{\Delta x}{3} (y_0 + 4y_1 + y_2).$$

We consider the definite integral

$$\int_a^b f(x)\,dx.$$

We assume that $f(x)$ is continuous on $[a, b]$ and we divide $[a, b]$ into an even number n of subintervals of equal length

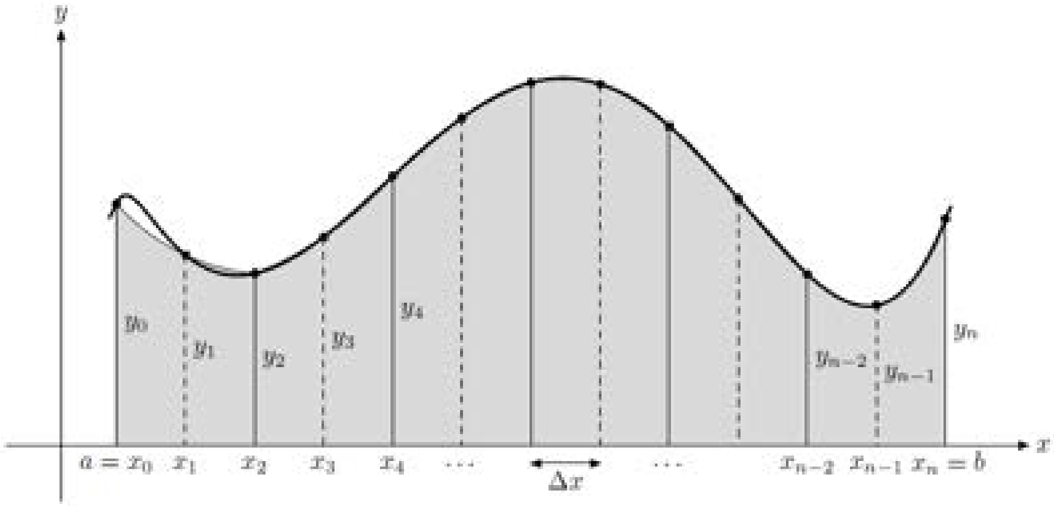
$$\Delta x = \frac{b-a}{n}$$

using the $n+1$ points

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \dots, \quad x_n = a + n\Delta x = b.$$

We can compute the value of $f(x)$ at these points.

$$y_0 = f(x_0), \quad y_1 = f(x_1), \quad y_2 = f(x_2), \quad \dots, \quad y_n = f(x_n).$$



We can estimate the integral by adding the areas under the parabolic arcs through three successive points.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + y_2) + \frac{\Delta x}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{\Delta x}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

By simplifying, we obtain Simpson's rule formula.

$$\boxed{\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n)}$$

Example. Use Simpson's rule with $n = 6$ to estimate

$$\int_1^4 \sqrt{1+x^3}\,dx.$$

Solution. For $n = 6$, we have $\Delta x = \frac{4-1}{6} = 0.5$. We compute the values of $y_0, y_1, y_2, \dots, y_6$.

x	1	1.5	2	2.5	3	3.5	4
$y = \sqrt{1+x^3}$	$\sqrt{2}$	$\sqrt{4.375}$	3	$\sqrt{16.625}$	$\sqrt{28}$	$\sqrt{43.875}$	$\sqrt{65}$

Therefore,

$$\begin{aligned}\int_1^4 \sqrt{1+x^3}\,dx &\approx \frac{0.5}{3}\left(\sqrt{2} + 4\sqrt{4.375} + 2(3) + 4\sqrt{16.625} + 2\sqrt{28} + 4\sqrt{43.875} + \sqrt{65}\right)\\ &\approx \boxed{12.871}\end{aligned}$$

Year 2: A Level Mathematics	
<u>Differentiation: Chain Rule</u>	
Self-Assessment: Please identify areas in which you believe are your strong points and those you feel you need to improve on Provide evidence to support your assessment with reference to the content in this booklet.	
Strengths	**Areas for Improvement**

The Chain Rule

mc-TY-chain-2009-1

A special rule, **the chain rule**, exists for differentiating a function of another function. This unit illustrates this rule.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- explain what is meant by a function of a function
- state the chain rule
- differentiate a function of a function

Contents

1. Introduction

In this unit we learn how to differentiate a 'function of a function'. We first explain what is meant by this term and then learn about the Chain Rule which is the technique used to perform the differentiation.

2. A function of a function

Consider the expression $\cos x^2$. Immediately we note that this is different from the straightforward cosine function, $\cos x$. We are finding the cosine of x^2, not simply the cosine of x. We call such an expression a 'function of a function'.

Suppose, in general, that we have two functions, $f(x)$ and $g(x)$. Then

$$y = f(g(x))$$

is a function of a function. In our case, the function f is the cosine function and the function g is the square function. We could identify them more mathematically by saying that

$$f(x) = \cos x \qquad g(x) = x^2$$

so that

$$f(g(x)) = f(x^2) = \cos x^2$$

Now let's have a look at another example. Suppose this time that f is the square function and g is the cosine function. That is,

$$f(x) = x^2 \qquad g(x) = \cos x$$

then

$$f(g(x)) = f(\cos x) = (\cos x)^2$$

We often write $(\cos x)^2$ as $\cos^2 x$. So $\cos^2 x$ is also a function of a function.

In the following section we learn how to differentiate such a function.

3. The chain rule

In order to differentiate a function of a function, $y = f(g(x))$, that is to find $\frac{\mathrm{d}y}{\mathrm{d}x}$, we need to do two things:

1. Substitute $u = g(x)$. This gives us

$$y = f(u)$$

Next we need to use a formula that is known as the Chain Rule.

2. **Chain Rule**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

Key Point

Chain rule:

To differentiate $y = f(g(x))$, let $u = g(x)$. Then $y = f(u)$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

Example

Suppose we want to differentiate $y = \cos x^2$.

Let $u = x^2$ so that $y = \cos u$.

It follows immediately that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x \qquad \frac{\mathrm{d}y}{\mathrm{d}u} = -\sin u$$

The chain rule says

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

and so

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= -\sin u \times 2x \\ &= -2x \sin x^2 \end{aligned}$$

Example

Suppose we want to differentiate $y = \cos^2 x = (\cos x)^2$.

Let $u = \cos x$ so that $y = u^2$

It follows that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \qquad \frac{\mathrm{d}y}{\mathrm{d}u} = 2u$$

Then

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= 2u \times -\sin x \\ &= -2 \cos x \sin x \end{aligned}$$

Example

Suppose we wish to differentiate $y = (2x - 5)^{10}$.

Now it might be tempting to say 'surely we could just multiply out the brackets'. To multiply out the brackets would take a long time and there are lots of opportunities for making mistakes. So let us treat this as a function of a function.

Let $u = 2x - 5$ so that $y = u^{10}$. It follows that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}u} = 10u^9$$

Then

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= 10u^9 \times 2 \\ &= 20(2x-5)^9\end{aligned}$$

4. Some examples involving trigonometric functions

In this section we consider a trigonometric example and develop it further to a more general case.

Example

Suppose we wish to differentiate $y = \sin 5x$.

Let $u = 5x$ so that $y = \sin u$. Differentiating

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 5 \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}u} = \cos u$$

From the chain rule

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= \cos u \times 5 \\ &= 5\cos 5x\end{aligned}$$

Notice how the 5 has appeared at the front, - and it does so because the derivative of $5x$ was 5. So the question is, could we do this with any number that appeared in front of the x, be it 5 or 6 or $\frac{1}{2}$, 0.5 or for that matter n ?

So let's have a look at another example.

Example

Suppose we want to differentiate $y = \sin nx$.

Let $u = nx$ so that $y = \sin u$. Differentiating

$$\frac{\mathrm{d}u}{\mathrm{d}x} = n \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}u} = \cos u$$

Quoting the formula again:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

So

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos u \times n \\ &= n\cos nx\end{aligned}$$

So the n's have behaved in exactly the same way that the 5's behaved in the previous example.

Key Point

$$\text{if } y = \sin nx \qquad \text{then} \qquad \frac{dy}{dx} = n\cos nx$$

For example, suppose $y = \sin 6x$ then $\frac{dy}{dx} = 6\cos 6x$ just by using the standard result.

Similar results follow by differentiating the cosine function:

Key Point

$$\text{if } y = \cos nx \qquad \text{then} \qquad \frac{dy}{dx} = -n\sin nx$$

So, for example, if $y = \cos\frac{1}{2}x$ then $\frac{dy}{dx} = -\frac{1}{2}\sin\frac{1}{2}x$.

5. A simple technique for differentiating directly

In this section we develop, through examples, a further result.

Example

Suppose we want to differentiate $y = e^{x^3}$.

Let $u = x^3$ so that $y = e^u$. Differentiating

$$\frac{du}{dx} = 3x^2 \qquad\qquad \frac{dy}{du} = e^u$$

Quoting the formula again:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

So

$$\begin{aligned}\frac{dy}{dx} &= e^u \times 3x^2 \\ &= 3x^2 e^{x^3}\end{aligned}$$

We will now explore how this relates to a general case, that of differentiating $y = f(g(x))$.
To differentiate $y = f(g(x))$, we let $u = g(x)$ so that $y = f(u)$.

The chain rule states

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

In what follows it will be convenient to reverse the order of the terms on the right:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} \times \frac{\mathrm{d}y}{\mathrm{d}u}$$

which, in terms of f and g we can write as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{d}{dx}(g(x)) \times \frac{d}{du}(f(g((x)))$$

This gives us a simple technique which, with some practice, enables us to apply the chain rule directly

Key Point

(i) given $y = f(g(x))$, identify the functions $f(u)$ and $g(x)$ where $u = g(x)$.

(ii) differentiate g and multiply by the derivative of f
where it is understood that the argument of f is $u = g(x)$.

Example

To differentiate $y = \tan x^2$ we apply these two stages:

(i) first identify $f(u)$ and $g(x)$: $f(u) = \tan u$ and $g(x) = x^2$.

(ii) differentiate $g(x)$: $\dfrac{\mathrm{d}g}{\mathrm{d}x} = 2x$. Multiply by the derivative of $f(u)$, which is $\sec^2 u$ to give

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \sec^2 x^2$$

Example

To differentiate $y = \mathrm{e}^{1+x^2}$.

(i) first identify $f(u)$ and $g(x)$: $f(u) = \mathrm{e}^u$ and $g(x) = 1 + x^2$.

(ii) differentiate $g(x)$: $\dfrac{\mathrm{d}g}{\mathrm{d}x} = 2x$. Multiply by the derivative of $f(u)$, which is e^u to give

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\,\mathrm{e}^{1+x^2}$$

You should be able to verify the remaining examples purely by inspection. Try it!

Example

$$\begin{aligned} y &= \sin(x + \mathrm{e}^x) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= (1 + \mathrm{e}^x)\cos(x + \mathrm{e}^x) \end{aligned}$$

Example

$$\begin{aligned} y &= \tan(x^2 + \sin x) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= (2x + \cos x).\sec^2(x^2 + \sin x) \end{aligned}$$

Example

$$\begin{aligned} y &= (2 - x^5)^9 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -5x^4.\ 9(2 - x^5)^8 \\ &= -45x^4(2 - x^5)^8 \end{aligned}$$

Example

$$\begin{aligned} y &= \ln(x + \sin x) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= (1 + \cos x)\,.\,\frac{1}{x + \sin x} \\ &= \frac{1 + \cos x}{x + \sin x} \end{aligned}$$

Exercises

1. Find the derivative of each of the following:
   a) $(3x - 7)^{12}$ b) $\sin(5x + 2)$ c) $\ln(2x - 1)$ d) e^{2-3x}
   e) $\sqrt{5x - 3}$ f) $(6x + 5)^{5/3}$ g) $\dfrac{1}{(3 - x)^4}$ h) $\cos(1 - 4x)$
2. Find the derivative of each of the following:
   a) $\ln(\sin x)$ b) $\sin(\ln x)$ c) $\mathrm{e}^{-\cos x}$ d) $\cos(\mathrm{e}^{-x})$
   e) $(\sin x + \cos x)^3$ f) $\sqrt{1 + x^2}$ g) $\dfrac{1}{\cos x}$ h) $\dfrac{1}{x^2 + 2x + 1}$

3. Find the derivative of each of the following:

a) $\ln\left(\sin^2 x\right)$ b) $\sin^2(\ln x)$ c) $\sqrt{\cos(3x-1)}$ d) $\left[1+\cos\left(x^2-1\right)\right]^{3/2}$

Answers

1. a) $36(3x-7)^{11}$ b) $5\cos(5x+2)$ c) $\dfrac{2}{2x-1}$ d) $-3\mathrm{e}^{2-3x}$

   e) $\dfrac{5}{2\sqrt{5x-3}}$ f) $10(6x+5)^{2/3}$ g) $\dfrac{4}{(3-x)^5}$ h) $4\sin(1-4x)$

2. a) $\dfrac{\cos x}{\sin x} = \cot x$ b) $\dfrac{\cos(\ln x)}{x}$ c) $\sin x\mathrm{e}^{-\cos x}$

   d) $\mathrm{e}^{-x}\sin\left(\mathrm{e}^{-x}\right)$ e) $3(\cos x - \sin x)(\sin x + \cos x)^2$ f) $\dfrac{x}{\sqrt{1+x^2}}$

   g) $\dfrac{\sin x}{\cos^2 x} = \tan x \sec x$ h) $\dfrac{-2(x+1)}{(x^2+2x+1)^4} = \dfrac{-2}{(x+1)^3}$

3. a) $\dfrac{2\cos x}{\sin x} = 2\cot x$ b) $\dfrac{2\sin(\ln x)\cos(\ln x)}{x}$

   c) $\dfrac{-3\sin(3x-1)}{2\sqrt{\cos(3x-1)}}$ d) $-3x\sin\left(x^2-1\right)\left[1+\cos\left(x^2-1\right)\right]^{1/2}$

Year 2: A Level Mathematics	
<u>Differentiation: Product Rule and Quotient Rule</u>	
Self-Assessment: Please identify areas in which you believe are your strong points and those you feel you need to improve on Provide evidence to support your assessment with reference to the content in this booklet.	
Strengths	**Areas for Improvement**

Engineering Maths First Aid Kit

8.4

Product and quotient rules

Introduction

As their names suggest, the **product rule** and the **quotient rule** are used to differentiate products of functions and quotients of functions. This leaflet explains how.

1.The product rule

It is appropriate to use this rule when you want to differentiate two functions which are multiplied together. For example

$$y = \mathrm{e}^x \sin x \qquad \text{is a product of the functions } \mathrm{e}^x \text{ and } \sin x$$

In the rule which follows we let u stand for the first of the functions and v stand for the second.

If u and v are functions of x, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

Example

If $y = 7x\mathrm{e}^{2x}$ find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

Comparing the given function with the product rule we let

$$u = 7x, \qquad v = \mathrm{e}^{2x}$$

It follows that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 7, \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 2\mathrm{e}^{2x}$$

Thus, using the product rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}(7x\mathrm{e}^{2x}) = 7x(2\mathrm{e}^{2x}) + \mathrm{e}^{2x}(7) = 7\mathrm{e}^{2x}(2x+1)$$

2. The quotient rule

It is appropriate to use this rule when you want to differentiate a quotient of two functions, that is, one function divided by another. For example

$$y = \frac{\mathrm{e}^x}{\sin x} \qquad \text{is a quotient of the functions } \mathrm{e}^x \text{ and } \sin x$$

In the rule which follows we let u stand for the function in the numerator and v stand for the function in the denominator.

If u and v are functions of x, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{du}{\mathrm{d}x} - u\frac{dv}{\mathrm{d}x}}{v^2}$$

Example

If $y = \dfrac{\sin x}{3x^2}$ find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

Comparing the given function with the quotient rule we let

$$u = \sin x, \qquad \text{and} \qquad v = 3x^2$$

It follows that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 6x$$

Applying the quotient rule gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2\cos x - \sin x\,(6x)}{9x^4} = \frac{3x(x\cos x - 2\sin x)}{9x^4} = \frac{x\cos x - 2\sin x}{3x^3}$$

Exercises

Choose an appropriate rule in each case to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

1. $y = x^2\sin x$
2. $y = \mathrm{e}^x\cos x$
3. $y = \frac{\mathrm{e}^x}{x^2+1}$
4. $y = \frac{x^2+1}{\mathrm{e}^x}$
5. $y = 7x\log_{\mathrm{e}} x$
6. $y = \frac{x-1}{\sin 2x}$

Answers

1. $x^2\cos x + 2x\sin x$ 2. $-\mathrm{e}^x\sin x + \mathrm{e}^x\cos x = \mathrm{e}^x(\cos x - \sin x)$ 3. $\frac{\mathrm{e}^x(x^2-2x+1)}{(x^2+1)^2}$

4. $\frac{2x-x^2-1}{\mathrm{e}^x}$, 5. $7(1+\log_{\mathrm{e}} x)$, 6. $\frac{\sin 2x - 2(x-1)\cos 2x}{\sin^2 2x}$.

Year 2: A Level Mathematics	
<u>Trigonometry: Identities</u>	
Self-Assessment: Please identify areas in which you believe are your strong points and those you feel you need to improve on Provide evidence to support your assessment with reference to the content in this booklet.	
Strengths	**Areas for Improvement**

USEFUL TRIGONOMETRIC IDENTITIES

Definitions

$$\tan x = \frac{\sin x}{\cos x}$$

$$\sec x = \frac{1}{\cos x} \qquad \operatorname{cosec} x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}$$

Fundamental trig identity

$$(\cos x)^2 + (\sin x)^2 = 1$$

$$1 + (\tan x)^2 = (\sec x)^2$$

$$(\cot x)^2 + 1 = (\operatorname{cosec} x)^2$$

You can get this one from the top one if you divide by $(\cos x)^2$

You can get this one from the top one if you divide by $(\sin x)^2$

Odd and even properties

$$\cos(-x) = \cos(x) \qquad \sin(-x) = -\sin(x) \qquad \tan(-x) = -\tan(x)$$

cos is an even function

sin and tan are odd functions

Double angle formulas

$$\sin(2x) = 2\sin x \cos x \qquad \cos(2x) = (\cos x)^2 - (\sin x)^2$$

$$\cos(2x) = 2(\cos x)^2 - 1$$

$$\cos(2x) = 1 - 2(\sin x)^2$$

You get this from the top one if you sub in $(\cos x)^2 = 1-(\sin x)^2$

You get this from the top one if you sub in $(\sin x)^2 = 1-(\cos x)^2$

Half angle formulas

$$\left[\sin(\tfrac{1}{2}x)\right]^2 = \tfrac{1}{2}(1 - \cos x)$$

$$\left[\cos(\tfrac{1}{2}x)\right]^2 = \tfrac{1}{2}(1 + \cos x)$$

These come from rearranging the cos double angle formulas and then replacing x with $\frac{1}{2}x$

Sums and differences of angles

If you sub $A=x$, $B=x$ into these you get the double angle formulas

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

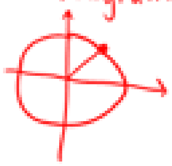
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

You can get these by replacing B with $-B$ in the other two and using the odd and even properties

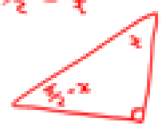
**** See other side for more identities ****

USEFUL TRIGONOMETRIC IDENTITIES

You can find these by drawing a diagram of the unit circle



You can find these by drawing a right-angled triangle with small angles x and $\pi/2 - x$



These are a combination of the above two sets of formulas and the odd/even properties

$$\begin{aligned}
\cos(x) &= \cos(x) \\
\cos(x+\tfrac{\pi}{2}) &= -\sin(x) \\
\cos(x+\pi) &= -\cos(x) \\
\cos(x+\tfrac{3\pi}{2}) &= \sin(x) \\
\cos(x+2\pi) &= \cos(x)
\end{aligned}
\qquad
\begin{aligned}
\cos(x) &= \cos(x) \\
\cos(x-\tfrac{\pi}{2}) &= \sin(x) \\
\cos(x-\pi) &= -\cos(x) \\
\cos(x-\tfrac{3\pi}{2}) &= -\sin(x) \\
\cos(x-2\pi) &= \cos(x)
\end{aligned}
\qquad
\begin{aligned}
\cos(-x) &= \cos(x) \\
\cos(\tfrac{\pi}{2}-x) &= \sin(x) \\
\cos(\pi-x) &= -\cos(x) \\
\cos(\tfrac{3\pi}{2}-x) &= -\sin(x) \\
\cos(2\pi-x) &= \cos(x)
\end{aligned}$$

$$\begin{aligned}
\sin(x) &= \sin(x) \\
\sin(x+\tfrac{\pi}{2}) &= \cos(x) \\
\sin(x+\pi) &= -\sin(x) \\
\sin(x+\tfrac{3\pi}{2}) &= -\cos(x) \\
\sin(x+2\pi) &= \sin(x)
\end{aligned}
\qquad
\begin{aligned}
\sin(x) &= \sin(x) \\
\sin(x-\tfrac{\pi}{2}) &= -\cos(x) \\
\sin(x-\pi) &= -\sin(x) \\
\sin(x-\tfrac{3\pi}{2}) &= \cos(x) \\
\sin(x-2\pi) &= \sin(x)
\end{aligned}
\qquad
\begin{aligned}
\sin(-x) &= -\sin(x) \\
\sin(\tfrac{\pi}{2}-x) &= \cos(x) \\
\sin(\pi-x) &= \sin(x) \\
\sin(\tfrac{3\pi}{2}-x) &= -\cos(x) \\
\sin(2\pi-x) &= -\sin(x)
\end{aligned}$$

$$\begin{aligned}
\tan(x) &= \tan(x) \\
\tan(x+\tfrac{\pi}{2}) &= -\cot(x) \\
\tan(x+\pi) &= \tan(x) \\
\tan(x+\tfrac{3\pi}{2}) &= -\cot(x) \\
\tan(x+2\pi) &= \tan(x)
\end{aligned}
\qquad
\begin{aligned}
\tan(x) &= \tan(x) \\
\tan(x-\tfrac{\pi}{2}) &= -\cot(x) \\
\tan(x-\pi) &= \tan(x) \\
\tan(x-\tfrac{3\pi}{2}) &= -\cot(x) \\
\tan(x-2\pi) &= \tan(x)
\end{aligned}
\qquad
\begin{aligned}
\tan(-x) &= -\tan(x) \\
\tan(\tfrac{\pi}{2}-x) &= \cot(x) \\
\tan(\pi-x) &= -\tan(x) \\
\tan(\tfrac{3\pi}{2}-x) &= \cot(x) \\
\tan(2\pi-x) &= -\tan(x)
\end{aligned}$$

**** See other side for more identities ****

TRIGONOMETRY
LAWS AND IDENTITIES

TANGENT IDENTITIES

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

RECIPROCAL IDENTITIES

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

PYTHAGOREAN IDENTITIES

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

PERIODIC IDENTITIES

$$\sin(\theta + 2\pi n) = \sin\theta$$

$$\cos(\theta + 2\pi n) = \cos\theta$$

$$\tan(\theta + \pi n) = \tan\theta$$

$$\csc(\theta + 2\pi n) = \csc\theta$$

$$\sec(\theta + 2\pi n) = \sec\theta$$

$$\cot(\theta + \pi n) = \cot\theta$$

EVEN/ODD IDENTITIES

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta$$

$$\sec(-\theta) = \sec\theta$$

$$\cot(-\theta) = -\cot\theta$$

DOUBLE ANGLE IDENTITIES

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

HALF ANGLE IDENTITIES

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

LAW OF COSINES

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

PRODUCT TO SUM IDENTITIES

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

SUM TO PRODUCT IDENTITIES

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

LAW OF SINES

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

LAW OF TANGENTS

$$\frac{a-b}{a+b} = \frac{\tan\left[\frac{1}{2}(\alpha-\beta)\right]}{\tan\left[\frac{1}{2}(\alpha+\beta)\right]}$$

$$\frac{b-c}{b+c} = \frac{\tan\left[\frac{1}{2}(\beta-\gamma)\right]}{\tan\left[\frac{1}{2}(\beta+\gamma)\right]}$$

$$\frac{a-c}{a+c} = \frac{\tan\left[\frac{1}{2}(\alpha-\gamma)\right]}{\tan\left[\frac{1}{2}(\alpha+\gamma)\right]}$$

SUM/DIFFERENCES IDENTITIES

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

MOLLWEIDE'S FORMULA

$$\frac{a+b}{c} = \frac{\cos\left[\frac{1}{2}(\alpha-\beta)\right]}{\sin\left(\frac{1}{2}\gamma\right)}$$

COFUNCTION IDENTITIES

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$

$$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

Trigonometric Equations and Identities Exam Questions (From OCR 4722)

Q1, (Jun 2012, Q7a)

(i) Given that α is the acute angle such that $\tan\alpha = \frac{2}{5}$, find the exact value of $\cos\alpha$. **[2]**

(ii) Given that β is the obtuse angle such that $\sin\beta = \frac{3}{7}$, find the exact value of $\cos\beta$. **[3]**

Q2, (OCR 4752, Jun 2006, Q3)

θ is an acute angle and $\sin\theta = \frac{1}{4}$. Find the exact value of $\tan\theta$. [3]

Q3, (OCR 4752, Jan 2007, Q3)

Given that $\cos\theta = \frac{1}{3}$ and θ is acute, find the exact value of $\tan\theta$. [3]

Q4, (OCR 4752, Jan 2008, Q3)

You are given that $\tan\theta = \frac{1}{2}$ and the angle θ is acute. Show, without using a calculator, that $\cos^2\theta = \frac{4}{5}$. [3]

Q5, (Jan 2010, Q1)

(i) Show that the equation

$$2\sin^2 x = 5\cos x - 1$$

can be expressed in the form

$$2\cos^2 x + 5\cos x - 3 = 0.$$

[2]

(ii) Hence solve the equation

$$2\sin^2 x = 5\cos x - 1,$$

giving all values of x between 0° and 360°. **[4]**

Q6, (Jun 2010, Q7)

(i) Show that $\dfrac{\sin^2 x - \cos^2 x}{1 - \sin^2 x} \equiv \tan^2 x - 1$. **[2]**

(ii) Hence solve the equation

$$\frac{\sin^2 x - \cos^2 x}{1 - \sin^2 x} = 5 - \tan x,$$

for $0° \leqslant x \leqslant 360°$. **[6]**

Q7, (Jun 2013, Q2)

Solve each of the following equations, for $0° \leqslant x \leqslant 360°$.

(i) $\sin\frac{1}{2}x = 0.8$ **[3]**

(ii) $\sin x = 3\cos x$ **[3]**

Q8, (Jun 2009, Q5)

Solve each of the following equations for $0° \leqslant x \leqslant 180°$.

(i) $\sin 2x = 0.5$ **[3]**

(ii) $2\sin^2 x = 2 - \sqrt{3}\cos x$ **[5]**

Q9, (Jun 2014, Q4)

(i) Show that the equation

$$\sin x - \cos x = \frac{6\cos x}{\tan x}$$

can be expressed in the form

$$\tan^2 x - \tan x - 6 = 0.$$ **[2]**

(ii) Hence solve the equation $\sin x - \cos x = \frac{6\cos x}{\tan x}$ for $0° \leqslant x \leqslant 360°$. **[4]**

Q10, (Jun 2017, Q9) [Modified]

The cubic polynomial f(x) is defined by $f(x) = 4x^3 + 9x - 5$.

(i) Show that $(2x-1)$ is a factor of f(x) and hence express f(x) as the product of a linear factor and a quadratic factor. **[4]**

(ii) **(a)** Show that the equation

$$4\sin 2\theta \cos 2\theta + \frac{5}{\cos 2\theta} = 13\tan 2\theta$$

can be expressed in the form

$$4\sin^3 2\theta + 9\sin 2\theta - 5 = 0.$$ **[4]**

(b) Hence solve the equation

$$4\sin 2\theta \cos 2\theta + \frac{5}{\cos 2\theta} = 13\tan 2\theta$$

for $0 \leqslant \theta \leqslant 360$ Give each answer in an exact form. **[4]**

Q11, (OCR 4752, Jun 2009, Q7)

Show that the equation $4\cos^2\theta = 4 - \sin\theta$ may be written in the form

$$4\sin^2\theta - \sin\theta = 0.$$

Hence solve the equation $4\cos^2\theta = 4 - \sin\theta$ for $0° \leqslant \theta \leqslant 180°$. **[5]**

Q12, (OCR 4752, Jun 2011, Q7)

Solve the equation $\tan\theta = 2\sin\theta$ for $0° \leqslant \theta \leqslant 360°$. **[4]**

Q13, (Jan 2008, Q9)

(i)

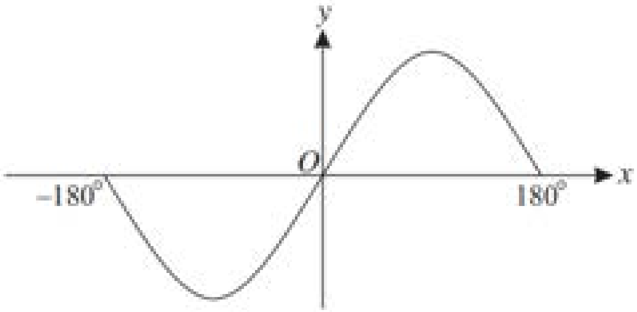


Fig. 1

Fig. 1 shows the curve $y = 2\sin x$ for values of x such that $-180° \leqslant x \leqslant 180°$. State the coordinates of the maximum and minimum points on this part of the curve. [2]

(ii)

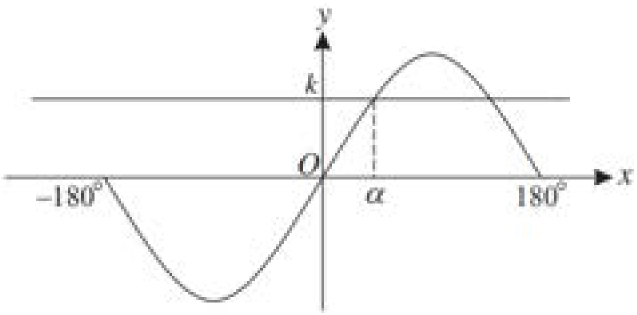


Fig. 2

Fig. 2 shows the curve $y = 2\sin x$ and the line $y = k$. The smallest positive solution of the equation $2\sin x = k$ is denoted by α. State, in terms of α, and in the range $-180° \leqslant x \leqslant 180°$,

(a) another solution of the equation $2\sin x = k$, [1]

(b) one solution of the equation $2\sin x = -k$. [1]

Q14, (OCR 4752, Jun 2013, Q9)

(i) Show that the equation $\dfrac{\tan\theta}{\cos\theta} = 1$ may be rewritten as $\sin\theta = 1 - \sin^2\theta$. **[2]**

(ii) Hence solve the equation $\dfrac{\tan\theta}{\cos\theta} = 1$ for $0° \leqslant \theta \leqslant 360°$. **[3]**

Q15, (Jun 2014, Q4)

(i) Show that the equation

$$\sin x - \cos x = \frac{6\cos x}{\tan x}$$

can be expressed in the form

$$\tan^2 x - \tan x - 6 = 0.$$ **[2]**

(ii) Hence solve the equation $\sin x - \cos x = \dfrac{6\cos x}{\tan x}$ for $0° \leqslant x \leqslant 360°$. **[4]**

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1 (a) Use the identity $\cos^2\theta + \sin^2\theta = 1$ to prove that $\tan^2\theta = \sec^2\theta - 1$ **(2)**

(b) Solve, for $0 \leq \theta \leq 360$, the equation,

$$\tan^2\theta + \sec^2\theta + 5\sec\theta = 2$$

Give your answers to 1 decimal place. **(5)**

(Total for question 1 is 7 marks)

---

2 (a) Use the identity $\cos^2\theta + \sin^2\theta = 1$ to prove that $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ **(2)**

(b) Solve, for $0 \leq \theta \leq 2\pi$, the equation,

$$\operatorname{cosec}^2\theta + \cot^2\theta = 3$$

Give your answers in terms of π. **(5)**

(Total for question 2 is 7 marks)

---

3 Solve, for $0 \leq x \leq 360$, the equation,

$$\tan^2 x + 4\sec x - 2 = 0$$

Give your answers to 1 decimal place.

(Total for question 3 is 5 marks)

---

4 Solve, for $-180 \leq x \leq 180$, the equation,

$$2\cot^2 x - \operatorname{cosec}^2 x + \operatorname{cosec} x = 4$$

Give your answers to 1 decimal place where appropriate.

(Total for question 4 is 5 marks)

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5 Prove the identities:

(a) $\sec^2 x - \operatorname{cosec}^2 x \equiv \tan^2 x - \cot^2 x$ **(2)**

(b) $(\sec x - \cos x)^2 \equiv \tan^2 x - \sin^2 x$ **(2)**

(Total for question 5 is 5 marks)

---

6 Prove that:

(a) $\sec^4 x - \tan^4 x \equiv 1 + 2\tan^2 x$ **(2)**

(b) Hence solve, for $0 \leq x \leq 360$, the equation, **(4)**

$$\sec^4 x - \tan^4 x = 3$$

(Total for question 6 is 6 marks)

Year 2: A Level Mathematics	
<u>Sequences and Series</u>	
Self-Assessment: Please identify areas in which you believe are your strong points and those you feel you need to improve on Provide evidence to support your assessment with reference to the content in this booklet.	
Strengths	**Areas for Improvement**

Sequences and Series Cheat Sheet

A sequence is a list of terms. For example, 3, 6, 9, 12, 15, ...
A series is the sum of a list of terms. For example, 3 + 6 + 9 + 12 + 15 + ...
The terms of a sequence are separated by a comma, while with a series they are all added together.

Definitions

Here are some important definitions prefacing the content in this chapter:

- A sequence is increasing if each term is greater than the previous.
  e.g. 4, 9, 14, 19, ...
- A sequence is decreasing if each term is less than the previous.
  e.g. 5, 4, 3, 2, 1, ...
- A sequence is periodic if the terms repeat in a cycle; $u_{n+k} = u_n$ for some k, which is known as the order of the sequence. e.g. -3, 1, -3, 1, -3, ... is periodic with order 2.

Arithmetic sequences

An arithmetic sequence is one where there is a common difference between each term. Arithmetic sequences are of the form

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad \ldots$$

where a is the first term and d is the common difference.

- The n[th] term of an arithmetic series is given by: $\boldsymbol{u_n = a + (n-1)d}$

Arithmetic series

An arithmetic series is the sum of the terms of an arithmetic sequence.

- The sum of the first n terms of an arithmetic series is given by $\boldsymbol{S_n = \frac{n}{2}[2a + (n-1)d]}$ or $\boldsymbol{S_n = \frac{n}{2}(a + l)}$

where a is the first term, d is the common difference and l is the last term.

You need to be able to prove this result. Here is the proof:

Example 1: Prove that the sum of the first n terms of an arithmetic series is $S_n = \frac{n}{2}[2a + (n-1)d]$.

We start by writing the sum out normally [1], and then in reverse [2]:

[1] $\quad S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-2)d) + (a+(n-1)d)$

[2] $\quad S_n = (a+(n-1)d) + (a+(n-2)d) + \cdots + (a+2d) + (a+d) + a$

Adding [1] and [2] gives us:

[1] + [2]: $\quad 2S_n = n(2a + (n-1)d)$

$$\therefore S_n = \frac{n}{2}[2a + (n-1)d]$$

Factorising out S_n from the LHS and a from the RHS

Example 2: The fifth term of an arithmetic series is 33. The tenth term is 68. The sum of the first n terms is 2225. Show that $7n^2 + 3n - 4450 = 0$, and hence find the value of n.

$5th\ term\ is\ 33 \therefore a + 4d = 33$ [1] (n[th] term formula)

$10th\ term\ is\ 68 \therefore 68 = a + 9d$ [2] (n[th] term formula)

Solving [1] and [2] simultaneously, we find that $d = 7$ and $a = 5$.

Sum of first n terms is 2225 $\therefore \frac{n}{2}[2a + (n-1)d] = 2225$

$$\frac{n}{2}[10 + (n-1)(7)] = 2225$$

$$n(3 + 7n) = 2225(2)$$

$$7n^2 + 3n - 4450 = 0$$

To find the value of n, we just need to solve the quadratic. Using the quadratic formula, we find that $n = 25\ or\ n = -25.4$. Since the term number must be a positive integer, we can conclude that $n = 25$.

Geometric sequences

The defining feature of a geometric sequence is that you must multiply by a common ratio, r, to get from one term to the next. Geometric sequences are of the form

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad \ldots$$

where a is the first term in the sequence and r is the common ratio.

- The n[th] term of a geometric sequence is given by: $\boldsymbol{u_n = ar^{n-1}}$

It can help in many questions to use the fact that $\frac{u_{k+1}}{u_k} = \frac{u_{k+2}}{u_{k+1}} = r$. This is especially helpful when the terms of the sequence are given in terms of an unknown constant. Part a of example 4 highlights this.

Geometric series

A geometric series is the sum of the terms of a geometric sequence.

- The sum of the first n terms of a geometric series is given by:

$$\boldsymbol{S_n = \frac{a(1 - r^n)}{1 - r}}$$

by multiplying the top and bottom of the fraction by -1, we can also use

$$\boldsymbol{S_n = \frac{a(r^n - 1)}{r - 1}}$$

You need to be able to prove this result. Here is the proof:

Example 3: Prove that the sum of the first n terms of a geometric series is $S_n = \frac{a(1-r^n)}{1-r}$

$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ [1]

multiplying the sum by r $\quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$ [2]

Subtracting [2] from [1] $\quad S_n - rS_n = a - ar^n$

$\Rightarrow S_n(1 - r) = a(1 - r^n)$ $\quad$ Factoring out S_n and a

$\therefore S_n = \frac{a(1-r^n)}{1-r}$ $\quad$ Dividing by $1 - r$

Since division by zero is undefined, this formula is invalid when $r = 1$.

Sum to infinity

The sum to infinity of a geometric sequence is the sum of the first n terms as n approaches infinity. This does not exist for all geometric sequences. Let's look at two examples:

$$2 + 4 + 8 + 16 + 32 + \cdots$$

Each term is twice the previous (i.e. $r = 2$). The sum of such a series is not finite, since each term is bigger than the previous. This is known as a divergent sequence.

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Here, each term is half the previous (i.e. $r = \frac{1}{2}$). The sum of such a series is finite, since as n becomes large, the terms will tend to 0. This is known as a convergent sequence.

- A geometric sequence is convergent if and only if $|r| < 1$.

The sum to infinity of a geometric sequence only exists for convergent sequences, and is given by:

$$\boldsymbol{S_\infty = \frac{a}{1 - r}}$$

Example 4: The first three terms of a geometric series are (k-6), k, (2k+5), where k is a positive constant.

a) Show that $k^2 - 7k - 30 = 0$
b) Hence find the value of k.
c) Find the common ratio of this series and hence calculate the sum of the first 10 terms.
d) Find the sum to infinity for a series with first term k and common ratio $\frac{k-6}{k}$.

a) Using the fact that $\frac{u_{k+1}}{u_k} = \frac{u_{k+2}}{u_{k+1}} = r$	$\frac{k}{k-6} = \frac{2k+5}{k}$
Cross-multiplying and simplifying:	$\Rightarrow k^2 = (2k+5)(k-6)$ $\Rightarrow k^2 = 2k^2 - 7k - 30$ $\Rightarrow k^2 - 7k - 30 = 0$ as required.
b) Solving the quadratic:	$(k-10)(k+3) = 0 \quad \Rightarrow \quad k = 10, \quad k = -3$ Since we are told k is positive, we can conclude $k = 10$.
c) From part a, $\frac{u_{k+1}}{u_k} = r = \frac{10}{10-6} = \frac{5}{2}$	$S_{10} = \frac{a(1-r^{10})}{1-r} = \frac{4(1-(2.5)^{10})}{1-2.5} = 25428.6 \Rightarrow 25400\ to\ 3\ s.f.$
d) $a = 10\ and\ r = \frac{1}{\frac{5}{2}} = \frac{2}{5}$	$\therefore S_\infty = \frac{10}{1 - \frac{2}{5}} = \frac{50}{3}$

Recurrence relations

A recurrence relation is simply another way of defining a sequence. With recurrence relations, each term is given as a function of the previous. For example, $u_{n+1} = u_n + 4,\ u_1 = 1$ represents an arithmetic sequence with first term 1 and common difference 4. In order to generate a recurrence relation, you need to know the first term.

Example 5: The sequence with recurrence relation $u_{k+1} = pu_k + q, u_1 = 5$, where p is a constant and $q = 13$, is periodic with order 2. Find the value of p.

We know the order is 2. So if $u_1 = 5$, then $u_3 = 5$ too. Finding u_3:	$u_2 = pu_1 + 13 = 5p + 13$ $u_3 = pu_2 + 13 = p(5p + 13) + 13$
Equating to 5:	$p(5p + 13) + 13 = 5$
Simplifying:	$5p^2 + 13p + 8 = 0$
Solving the quadratic by factorising: We get 2 values, one of which is correct.	$(5p + 8)(p + 1) = 0$ $p = -1$ or $p = -1.6$
Substitute $p = -1.6$ and $p = -1$ into the recurrence relation separately to see which one correctly corresponds to a periodic sequence of order 2.	Substituting $p = -1.6$ into the recurrence relation gives a sequence where each term is 5 and so does not have order 2. Using $p = -1$ does give us a periodic sequence with order 2 however, so $p = -1$.

Sigma notation

You need to be comfortable solving problems where series are given in sigma notation. Below is an annotated example explaining how the sigma notation is used.

This tells us the last value of r for our sequence, i.e. our last term will be $7 - 2(20) = -33$

Inputting $r = 1$ into this expression gives us the first term, $r = 2$ gives us the second term, and so forth.

$$\sum_{r=1}^{20}(7 - 2r) = 5 + 3 + 1 - 1 + \cdots$$

This is the value of r where our series starts, i.e. our first term is $7 - 2(1) = 5$.

This series in particular represents an arithmetic series with first term 5 and common difference -2

If you are ever troubled by a series given in sigma notation, it is a good idea to write out the first few terms and analyse the series that way.

Modelling with series

Geometric and arithmetic sequences are often used to model real-life scenarios. Consider the amount of money in a savings account; this can be modelled by a geometric sequence where r represents the interest paid at the end of each year and a is the amount of money in the account at the time of opening.

You need to be able to apply your knowledge of sequences and series to questions involving real-life scenarios. It is important to properly understand the context given to you, so take some time to read through the question more than once.

Example 6: A virus is spreading such that the number of people infected increases by 4% each day. Initially 100 people were diagnosed with the virus. How many days will it be before 1000 are infected?

$a = 100$ and $r = 1.04$. We are really just trying to find the smallest value of n such that $U_n > 1000$. $\quad U_n = 1000 < 100(1.04^n)$

1. divide both side by 100 $\quad 10 < (1.04^n)$

2. take logs of both sides $\quad \log(10) < n \log(1.04)$

3. divide both side by log(1.04) to solve for n (note that log(10) = 1) $\quad \frac{\log(10)}{\log(1.04)} < n$

$58.7 \ldots < n$

4. round your answer up $\quad \therefore n = 59$

Paper Reference(s)

6663/01

Edexcel GCE

Core Mathematics C1

Advanced Subsidiary

Sequences and Series: Arithmetic Series

Calculators may NOT be used for these questions.

Information for Candidates

A booklet 'Mathematical Formulae and Statistical Tables' might be needed for some questions.

The marks for the parts of questions are shown in round brackets, e.g. **(2)**.

There are 18 questions in this test.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.

You must show sufficient working to make your methods clear.

Answers without working may not gain full credit.

1. A farmer has a pay scheme to keep fruit pickers working throughout the 30 day season. He pays £a for their first day, £($a + d$) for their second day, £($a + 2d$) for their third day, and so on, thus increasing the daily payment by £d for each extra day they work.

A picker who works for all 30 days will earn £40.75 on the final day.

(a) Use this information to form an equation in a and d.

(2)

A picker who works for all 30 days will earn a total of £1005

(b) Show that $15(a + 40.75) = 1005$

(2)

(c) Hence find the value of a and the value of d.

(4)
(Total 8 marks)

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2. Jill gave money to a charity over a 20-year period, from Year 1 to Year 20 inclusive. She gave £150 in Year 1, £160 in Year 2, £170 in Year 3, and so on, so that the amounts of money she gave each year formed an arithmetic sequence.

(a) Find the amount of money she gave in Year 10.

(2)

(b) Calculate the total amount of money she gave over the 20-year period.

(3)

Kevin also gave money to the charity over the same 20-year period.

He gave £A in Year 1 and the amounts of money he gave each year increased, forming an arithmetic sequence with common difference £30.

The total amount of money that Kevin gave over the 20-year period was **twice** the total amount of money that Jill gave.

(c) Calculate the value of A.

(4)
(Total 9 marks)

---

3. A 40-year building programme for new houses began in Oldtown in the year 1951 (Year 1) and finished in 1990 (Year 40).

The numbers of houses built each year form an arithmetic sequence with first term a and common difference d.

Given that 2400 new houses were built in 1960 and 600 new houses were built in 1990, find

(a) the value of d,

(3)

(b) the value of a,

(2)

(c) the total number of houses built in Oldtown over the 40-year period.

(3)
(Total 8 marks)

---

4. The first term of an arithmetic series is a and the common difference is d.

The 18th term of the series is 25 and the 21st term of the series is $32\frac{1}{2}$.

(a) Use this information to write down two equations for a and d.

(2)

(b) Show that $a = –7.5$ and find the value of d.

(2)

The sum of the first n terms of the series is 2750.

(c) Show that n is given by

$$n^2 - 15n = 55 \times 40.$$

(4)

(d) Hence find the value of n.

(3)
(Total 11 marks)

---

5. Sue is training for a marathon. Her training includes a run every Saturday starting with a run of 5 km on the first Saturday. Each Saturday she increases the length of her run from the previous Saturday by 2 km.

(a) Show that on the 4th Saturday of training she runs 11 km.

(1)

(b) Find an expression, in terms of n, for the length of her training run on the nth Saturday.

(2)

(c) Show that the total distance she runs on Saturdays in n weeks of training is $n(n + 4)$ km.

(3)

On the n th Saturday Sue runs 43 km.

(d) Find the value of n.

(2)

(e) Find the total distance, in km, Sue runs on Saturdays in n weeks of training.

(2)
(Total 10 marks)

---

6. The first term of an arithmetic sequence is 30 and the common difference is –1.5

(a) Find the value of the 25th term.

(2)

The rth term of the sequence is 0.

(b) Find the value of r.

(2)

The sum of the first n terms of the sequence is S_n.

(c) Find the largest positive value of S_n.

(3)
(Total 7 marks)

---

7. A girl saves money over a period of 200 weeks. She saves 5p in Week 1, 7p in Week 2, 9p in Week 3, and so on until Week 200. Her weekly savings form an arithmetic sequence.

(a) Find the amount she saves in Week 200.

(3)

(b) Calculate her total savings over the complete 200 week period.

(3)
(Total 6 marks)

---

8. Ann has some sticks that are all of the same length. She arranges them in squares and has made the following 3 rows of patterns:

Row 1 

Row 2 

Row 3 

She notices that 4 sticks are required to make the single square in the first row, 7 sticks to make 2 squares in the second row and in the third row she needs 10 sticks to make 3 squares.

(a) Find an expression, in terms of n, for the number of sticks required to make a similar arrangement of n squares in the nth row.

(3)

Ann continues to make squares following the same pattern. She makes 4 squares in the 4th row and so on until she has completed 10 rows.

(b) Find the total number of sticks Ann uses in making these 10 rows.

(3)

Ann started with 1750 sticks. Given that Ann continues the pattern to complete k rows but does not have sufficient sticks to complete the $(k+ 1)$th row,

(c) show that k satisfies $(3k – 100)(k + 35) < 0$.

(4)

(d) Find the value of k.

(2)

(Total 12 marks)

9. An athlete prepares for a race by completing a practice run on each of 11 consecutive days. On each day after the first day, he runs further than he ran on the previous day. The lengths of his 11 practice runs form an arithmetic sequence with first term a km and common difference d km.

He runs 9 km on the 11th day, and he runs a total of 77 km over the 11 day period.

Find the value of a and the value of d.

(Total 7 marks)

---

10. On Alice's 11th birthday she started to receive an annual allowance. The first annual allowance was £500 and on each following birthday the allowance was increased by £200.

(a) Show that, immediately after her 12th birthday, the total of the allowances that Alice had received was £1200.

(1)

(b) Find the amount of Alice's annual allowance on her 18th birthday.

(2)

(c) Find the total of the allowances that Alice had received up to and including her 18th birthday.

(3)

When the total of the allowances that Alice had received reached £32 000 the allowance stopped.

(d) Find how old Alice was when she received her last allowance.

(7)

(Total 13 marks)

---

11. An arithmetic series has first term a and common difference d.

(a) Prove that the sum of the first n terms of the series is

$$\tfrac{1}{2}n[2a + (n - 1)d].$$

(4)

The rth term of a sequence is $(5r - 2)$.

(b) Write down the first, second and third terms of this sequence.

(1)

(c) Show that $$\sum_{r=1}^{n}(5r - 2) = \frac{1}{2}n(5n + 1).$$

(3)

(d) Hence, or otherwise, find the value of $\sum_{r=5}^{200}(5r - 2)$.

(4)

(Total 12 marks)

---

12. An arithmetic series has first term a and common difference d.

(a) Prove that the sum of the first n terms of the series is

$$\tfrac{1}{2}n[2a + (n - 1)d].$$

(4)

Sean repays a loan over a period of n months. His monthly repayments form an arithmetic sequence.

He repays £149 in the first month, £147 in the second month, £145 in the third month, and so on. He makes his final repayment in the nth month, where $n > 21$.

(b) Find the amount Sean repays in the 21st month.

(2)

Over the n months, he repays a total of £5000.

(c) Form an equation in n, and show that your equation may be written as

$$n^2 - 150n + 5000 = 0.$$

(3)

(d) Solve the equation in part (c).

(3)

(e) State, with a reason, which of the solutions to the equation in part (c) is **not** a sensible solution to the repayment problem.

(1)
(Total 13 marks)

13. The rth term of an arithmetic series is $(2r - 5)$.

(a) Write down the first three terms of this series.

(2)

(b) State the value of the common difference.

(1)

(c) Show that $\sum_{r=1}^{n}(2r - 5) = n(n - 4)$.

(3)
(Total 6 marks)

---

14. The first three terms of an arithmetic series are p, $5p - 8$, and $3p + 8$ respectively.

(a) Show that $p = 4$.

(2)

(b) Find the value of the 40th term of this series.

(3)

(c) Prove that the sum of the first n terms of the series is a perfect square.

(3)
(Total 8 marks)

---

15. The sum of an arithmetic series is

$$\sum_{r=1}^{n}(80 - 3r)$$

(a) Write down the first two terms of the series.

(2)

(b) Find the common difference of the series.

(1)

Given that $n = 50$,

(c) find the sum of the series.

(3)
(Total 6 marks)

---

16. In the first month after opening, a mobile phone shop sold 280 phones. A model for future trading assumes that sales will increase by x phones per month for the next 35 months, so that $(280 + x)$ phones will be sold in the second month, $(280 + 2x)$ in the third month, and so on.

Using this model with $x = 5$, calculate

(a) (i) the number of phones sold in the 36th month,

(2)

(ii) the total number of phones sold over the 36 months.

(2)

The shop sets a sales target of 17 000 phones to be sold over the 36 months.

Using the same model,

(b) find the least value of x required to achieve this target.

(4)

(Total 8 marks)

---

17. (a) An arithmetic series has first term a and common difference d. Prove that the sum of the first n terms of this series is

$$\frac{1}{2}n[2a + (n - 1)d].$$

(4)

The first three terms of an arithmetic series are k, 7.5 and $k + 7$ respectively.

(b) Find the value of k.

(2)

(c) Find the sum of the first 31 terms of this series.

(4)

(Total 10 marks)

---

18. An arithmetic series has first term a and common difference d.

(a) Prove that the sum of the first n terms of the series is

$$\frac{1}{2}n[2a + (n - 1)d].$$

(4)

A polygon has 16 sides. The lengths of the sides of the polygon, starting with the shortest side, form an arithmetic sequence with common difference d cm.

The longest side of the polygon has length 6 cm and the perimeter of the polygon is 72 cm.

Find

(b) the length of the shortest side of the polygon,

(5)

(c) the value of d.

(2)
(Total 11 marks)

---

1. (a) $a + 29d = 40.75$ or $a = 40.75 - 29d$ or $29d = 40.75 - a$ M1 A1 2

Note

Parts (b) and (c) may run together

M1 for attempt to use $a + (n - 1)d$ with $n = 30$ to form an equation .
So $a + (30 - 1)d$ = any number is OK

A1 as written. Must see $29d$ not just $(30 - 1)d$.
Ignore any floating £ signs e.g. $a + 29d$ = £40.75 is OK for M1A1
These two marks must be scored in (a). Some may omit (a) but get correct equation in (c) [or (b)] but we do not give the marks retrospectively.

(b) $(S_{30}) = \frac{30}{2}(a + l)$ or $\frac{30}{2}(a + 40.75)$ or $\frac{30}{2}(2a + (30 - 1)d)$ or $15(2a + 29d)$ M1

So $1005 = 15[a + 40.75]$ * A1 cso 2

Note

Parts (b) and (c) may run together

M1 for an attempt to use an S_n formula with $n = 30$.

Must see one of the printed forms. (S_{30} = is not required)

A1cso for forming an equation with 1005 and S_n and simplifying to printed answer.
Condone £ signs e.g. 15[a + £40.75] = 1005 is OK for A1

(c) $67 = a + 40.75$ so $\underline{a = (£)\ 26.25 \text{ or } 2625\text{p or } 26\tfrac{1}{4} \text{ NOT } \tfrac{105}{4}}$ M1 A1

$29d = 40.75 - 26.25$ M1

$= 14.5$ so $\underline{d = (£)0.50 \text{ or } 0.5 \text{ or } 50\text{p}}$ or $\frac{1}{2}$ A1 4

Note

1st M1 for an attempt to simplify the given linear equation for a.
Correct processes. Must get to $ka = \ldots$ or $k = a + m$ i.e. one step (division or subtraction) from $a = \ldots$ Commonly:
$15a = 1005 - 611.25\ (= 393.75)$

1st A1 For $a = 26.25$ or 2625p or $26\frac{1}{4}$ NOT $\frac{105}{4}$ or any other fraction

2nd M1 for correct attempt at a linear equation for d, follow through their a or equation in (a) Equation just has to be linear in d, they don't have to simplify to $d = \ldots$

2nd A1 depends upon 2nd M1 and use of correct a. Do not penalise a second time if there were minor arithmetic errors in finding a provided $a = 26.25$ (o.e.) is used.

Do not accept other fractions other than $\frac{1}{2}$

If answer is in pence a "p" must be seen.

Sim Equ

Use this scheme: 1st M1A1 for *a* and 2nd M1A1 for *d*.
Typically solving: $1005=30a + 435d$ and $40.75 = a + 29d$.
If they find *d* first then follow through use of their *d* when finding *a*.

[8]

2. (a) $a + 9d = 150 + 9 \times 10 = 240$ M1 A1 2

Note

M: Using $a + 9d$ with at least one of $a = 150$ and $d = 10$.

Being 'one off' (e.g. equivalent to $a + 10d$), scores M0.

Correct answer with no working scores both marks.

'Listing' and other methods

M: Listing terms (found by a correct method with at least one of a = 150 and $d = 10$), and picking the 10th term. (There may be numerical slips).

(b) $\frac{1}{2}n\{2a+(n-1)d\} = \frac{20}{2}\{2\times150+19\times10\}, = 4900$ M1 A1, A1 3

Note

M: Attempting to use the correct sum formula to obtain S_{20}, with at least one of $a = 150$ and $d = 10$. If the wrong value of *n* or *a* or *d* is used, the M mark is only scored if the correct sum formula has been quoted.

1st A: Any fully correct numerical version.

'Listing' and other methods

M: Listing sums, or listing and adding terms (found by a correct method with at least one of a = 150 and d = 10), far enough to establish the required sum. (There may be numerical slips). Note: 20th term is 340.

A2 (scored as A1 A1) for 4900 (clearly selected as the answer).

If no working (or no legitimate working) is seen, but the answer 4900 is given, allow one mark (scored as M1 A0 A0).

(c) Kevin: $\frac{1}{2}n\{2a+(n-1)d\}=\frac{20}{2}\{2A+19\times 30\}$ B1

Kevin's total = 2×"4900" (or "4900" = 2× Kevin's total) M1

$\frac{20}{2}\{2A+19\times 30\}=2\times\text{"4900"}$ A1ft

$A = 205$ A1 4

<u>'Listing' and other methods</u>

<u>By trial and improvement:</u>

Obtaining a value of *A* for which Kevin's total is twice Jill's total, or Jill's total is twice Kevin's (using Jill's total from (b)): M1
Obtaining a value of *A* for which Kevin's total is twice Jill's total (using Jill's total from (b)): A1ft
Fully correct solutions then score the B1 and final A1.

The answer 205 with no working (or no legitimate working) scores no marks.

[9]

3. (a) $a + 9d = 2400$ $a + 39d = 600$ M1

$d = \frac{-1800}{30}$ $d = -60$ (accept ± 60 for A1) M1 A1 3

<u>Note</u>

If the sequence is considered 'backwards', an equivalent solution may be given using $d = 60$ with $a = 600$ and $l = 2940$ for part (b). This can still score full marks. **Ignore labelling of (a) and (b)**

1st M1 for an attempt to use 2400 and 600 in a+(n – 1)d formula. Must use both values

i.e. need $a + pd = 2400$ <u>and</u> $a + qd = 600$ where $p = 8$ or 9 and $q = 38$ or 39 (any combination)

2nd M1 for an attempt to solve <u>their</u> 2 linear equations in *a* and *d* as far as *d* =…

A1 for $d = \pm 60$. Condone correct equations leading to $d = 60$ or $a + 8d = 2400$ and $a + 38d = 600$ leading to $d = -60$. They should get penalised in (b) and (c).
NB This is a "one off" ruling for A1. Usually an A mark must follow from their work.

ALT

1st M1 for $(30d) = \pm(2400 - 600)$

2nd M1 for $(d =) \pm \frac{(2400-600)}{30}$

A1 for $d = \pm 60$

$a + 9d = 600$, $a + 39d = 2400$ **only** scores M0 BUT if they solve to find $d = \pm 60$ then use ALT scheme above.

(b) $a - 540 = 2400$ $a = 2940$ M1 A1 2

Note

M1 for use of their d in a **correct** linear equation to find a leading to $a = \ldots$

A1 their a must be compatible with their d so $d = 60$ must have $a = 600$ and $d = -60$, $a = 2940$

So for example they can have $2400 = a + 9(60)$ leading to $a = \ldots$ for M1 but it scores A0

Any approach using a list scores M1A1 for a correct a but M0A0 otherwise

(c) Total

$= \frac{1}{2}n\{2a + (n-1)d\} = \frac{1}{2} \times 40 \times (5880 + 39 \times -60)$(ft values of a and d) M1 A1ft

$= \underline{70800}$ A1cao 3

Note

M1 for use of a correct S_n formula with $n = 40$ and at least one of a, d or l correct or correct ft.

1st A1ft for use of a correct S_{40} formula and both a, d or a, l correct or correct follow through

ALT

Total $= \frac{1}{2}n\{a + l\} = \frac{1}{2} \times 40 \times (2940 + 600)$ (ft value of a) M1 A1ft

2nd A1 for 70800 only

[8]

4. (a) $a + 17d = 25$ or equiv. (for 1st B1),

$a + 20d = 32.5$ or equiv. (for 2nd B1), B1, B1 2

Note

Alternative:

1st B1: $d = 2.5$ or equiv.or $d = \frac{32.5 - 25}{3}$, No method required, but $a = -17.5$ must not be assumed.

2nd B1: Either $a + 17d = 25$ or $a + 20d = 32.5$ seen, or used with a value of d…
or for 'listing terms' or similar methods, 'counting back' 17 (or 20) terms.

(b) Solving (Subtract) $3d = 7.5$ so $\boldsymbol{d} = \underline{2.5}$ M1

$a = 32.5 - 20 \times 2.5$ so $\boldsymbol{a} = \underline{-17.5}$ (*) A1cso 2

Note

M1: In main scheme: for a full method (allow numerical or sign slips) leading to solution for d or a without assuming $a = -17.5$
In alternative scheme: for using a d value to find a value for a.

A1: Finding correct values for both a and d (allowing equiv. fractions such as $d = 15/6$), with no incorrect working seen.

(c) $2750 = \frac{n}{2}[-35 + \frac{5}{2}(n-1)]$ M1A1ft

{ $4 \times 2750 = n(5n - 75)$ }

$4 \times 550 = n(n - 15)$ M1

$\underline{n^2 - 15n = 55 \times 40}$ (*) A1cso 4

Note

In the main scheme, if the given a is used to find d from one of the equations, then allow M1A1 if both values are checked in the 2nd equation.

1st M1 for attempt to form equation with correct S_n formula and 2750, with values of a and d.

1st A1ft for a correct equation following through their d.

2nd M1 for expanding and simplifying to a 3 term quadratic.

2nd A1 for correct working leading to printed result (no incorrect working seen).

(d) $n^2 - 15n - 55 \times 40 = 0$ or $n^2 - 15n - 2200 = 0$ M1

$(n - 55)(n + 40) = 0$ $n = \ldots$ M1

$\boldsymbol{n = 55}$ (ignore – 40) A1 3

Note

1st M1 forming the correct 3TQ = 0 . Can condone missing "= 0" but all terms must be on one side.

First M1 can be implied (perhaps seen in (c), but there must be an attempt at (d) for it to be scored).

2nd M1 for attempt to solve 3TQ, by factorisation, formula or completing the square (see general marking principles at end of scheme). If this mark is earned for the 'completing the square' method or if the factors are written down directly, the 1st M1 is given by implication.

A1 for $n = 55$ dependent on both Ms. Ignore – 40 if seen.

No working or 'trial and improvement' methods in (d) score all 3 marks for the answer 55, otherwise no marks.

[11]

5. (a)	5, 7, 9, 11 or 5 + 2 + 2 + 2 = 11 or 5 + 6 = 11 use $a = 5$, $d = 2$, $n = 4$ and $t_4 = 5 + 3 \times 2 = 11$	B1	1
	B1 Any other sum must have a convincing argument		
(b)	$t_n = a + (n-1)d$ with one of $a = 5$ or $d = 2$ correct (can have a letter for the other)	M1	
	$= 5 + 2(n-1)$ or $2n + 3$ or $1 + 2(n+1)$	A1	2

M1 for an attempt to use $a + (n-1)d$ with one of a or d correct (the other can be a letter)
Allow any answer of the form $2n + p$ ($p \neq 5$) to score M1.

A1 for a correct expression (needn't be simplified)
[**Beware** $5 + (2n-1)$ scores A0]
Expression must be in n not x.
Correct answers with no working scores 2/2.

(c)	$S_n = \frac{n}{2}[2 \times 5 + 2(n-1)]$ or use of $\frac{n}{2}(5 + \text{“their } 2n+3\text{”})$ (may also be scored in (b))	M1A1	
	$= \{n(5 + n - 1)\} = n(n+4)$ (*)	A1cso	3

M1 for an attempt to use S_n formula with $a = 5$ or $d = 2$ or $a = 5$ and their "$2n + 3$"

1st A1 for a fully correct expression

2nd A1 for correctly simplifying to given answer.
No incorrect working seen. Must see S_n used.
Do not give credit for part (b) if the equivalent work is given in part (d)

(d)	$43 = 2n + 3$	M1	
	$[n] = 20$	A1	2

M1 for forming a suitable equation in n (ft their (b)) and attempting to solve leading to $n =$…

A1 for 20
Correct answer only scores 2/2. Allow 20 following a restart but check working.
eg $43 = 2n + 5$ that leads to $40 = 2n$ and $n =20$ should score M1A0.

(e)	$S_{20} = 20 \times 24$, $= \underline{480}$ (km)	M1A1	2

M1 for using their answer for n in $n(n+4)$ or S_n formula, their n must be a value.

A1 for 480 (ignore units but accept 480 000 m etc)
[no matter where their 20 comes from]

NB "attempting to solve" eg part (d) means we will allow sign slips and slips in arithmetic but not in processes. So dividing when they should subtract etc would lead to M0.
Listing in parts (d) and (e) can score 2 (if correct) or 0 otherwise in each part.
Poor labelling may occur (especially in (b) and (c)).
If you see work to get $n(n + 4)$ mark as (c)

[10]

6 (a) $u_{25} = a + 24d = 30 + 24 \times (-1.5)$ M1
$= -6$ A1 2

M: Substitution of $a = 30$ and $d = \pm 1.5$ into $(a + 24d)$.
Use of $a + 25d$ (or any other variations on 24) scores M0.

(b) $a + (n - 1)d = 30 - 1.5(r - 1) = 0$ M1
$r = 21$ A1 2

M: Attempting to use the term formula, equated to 0, to form an equation in r (with no other unknowns).
Allow this to be called n instead of r.
Here, being 'one off (e.g. equivalent to $a + nd$), scores M1.

(c) $S_{20} = \frac{20}{2}\{60 + 19(-1.5)\}$ or $S_{21} = \frac{21}{2}\{60 + 20(-1.5)\}$

or $S_{21} = \frac{21}{2}\{30 + 0\}$ M1A1ft

$= 315$ A1 3

M: Attempting to use the correct sum formula to obtain S_{20}, S_{21}, or, with their r from part (b), S_{r-1} or S_r.

1st A(ft): A correct numerical expression for S_{20}, S_{21}, or, with their r from part (b), S_{r-1} *or* S_r…. but the ft is dependent on an integer value of r.

Methods such as calculus to find a maximum only begin to score marks after establishing a value of r at which the maximum sum occurs. This value of r can be used for the M1 A1ft, but must be a positive integer to score A marks, so evaluation with, say, $n = 20.5$ would score M1 A0 A0.

<u>'Listing' and other methods</u>

(a) M: Listing terms (found by a correct method), and picking the <u>25</u>th term. (There may be numerical slips).

(b) M: Listing terms (found by a correct method), until the zero term is seen. (There may be numerical slips).
'Trial and error' approaches (or where working is unclear or non-existent) score M1 A1 for 21, M1 A0 for 20 or 22, and M0 A0 otherwise.

(c) M: Listing sums, or listing and adding terms (found by a correct method), at least as far as the 20th term. (There may be numerical slips).
A2 (scored as A1 A1) for 315 (clearly selected as the answer).
'Trial and error' approaches essentially follow the main scheme, beginning to score marks when trying S_{20}, S_{21}, or, with their r from part (b), S_{r-1} or S_r.

If no working (or no legitimate working) is seen, but the answer 315 is given, allow one mark (scored as M1 A0 A0).

<u>For reference</u>:

Sums: 30, 58.5, 85.5, 111, 135, 157.5, 178.5, 198, 216, 232.5, 247.5, 261, 273, 283.5, 292.5, 300, 306, 310.5, 313.5, 315, ...

[7]

7	(a)	Identify $a = 5$ and $d = 2$	May be implied	B1	
		$(u_{200} =)a + (200 – 1)d$ $(= 5 + (200 – 1) \times 2)$		M1	
		$=$ <u>430</u>(p) or (£)<u>4.30</u>		A1	3

B1 can be implied if the correct answer is obtained.
If 403 is <u>not</u> obtained then the values of a and d must be clearly identified as $a = 5$ and $d = 2$.
This mark can be awarded at any point.

M1 for attempt to use nth term formula with $n = 200$.
Follow through their a and d.
Must have use of $n = 200$ and one of a or d correct or correct follow through.
Must be 199 not 200.

A1 for 403 or 4.03 (i.e. condone missing £ sign here).
Condone £403 here.

NB $a = 3$, $d = 2$ is B0 and $a + 200d$ is M0 <u>BUT</u> $3 + 200 \times 2$ is B1M1 and A1 if it leads to 403.
Answer only of 403 (or 4.03)scores 3/3.

ALT <u>Listing</u>

They might score B1 if $a = 5$ and $d = 2$ are clearly identified.
Then award M1A1 together for 403.

(b) $(S_{200} =) \frac{200}{2}[2a + (200 - 1)d]$ or $\frac{200}{2}(a +$ "their 403") M1

$= \frac{200}{2}[2 \times 5 + (200 - 1) \times 2]$ or $\frac{200}{2}(5 +$ "their 403") A1

= <u>40 800</u> or <u>£408</u> A1 3

M1 for use of correct sum formula with $n = 200$.
Follow through their a and d and their 403.
Must have <u>some</u> use of $n = 200$, and some of a, d or l correct follow through.

1st A1 for any correct expression (i.e. must have $a = 5$ and $d = 2$) but can f.t. their 403 still.

2nd A1 for 40800 or £408 (i.e. £ sign is required before we accept 408 this time).
40800p is fine for A1 but £40800 is A0.

ALT <u>Listing</u>

$\sum_{r=1}^{200}(2r+3)$. Give M1 for $2 \times \frac{200}{2} \times (201) + 3k$ (with $k > 1$),
A1 for $k = 200$ and A1 for 40800.

[6]

8 (a) Recognising arithmetic series with first term 4 and common difference 3. B1
(If not scored here, this mark may be given if seen elsewhere in the solution).
$a + (n - 1)d = 4 + 3(n - 1)$ $(= 3n + 1)$ M1A1 3

B1: Usually identified by $a = 4$ and $d = 3$.

M1: Attempted use of term formula for arithmetic series, or… answer in the form ($3n$ + constant), where the constant is a non-zero value

Answer for (a) does not require simplification, and a correct answer without working scores all 3 marks.

(b) $S_n = \frac{n}{2}\{2a+(n-1)d\} = \frac{10}{2}\{8+(10-1)\times 3\}, = 175,$ M1A1, A1 3

M1: Use of correct sum formula with $n = 9$, 10 or 11.

A1: Correct, perhaps unsimplified, numerical version. A1: 175

<u>Alternative</u>: (Listing and summing terms).

M1: Summing 9, 10 or 11 terms. (At least 1st, 2nd and last terms must be seen).

A1: Correct terms (perhaps implied by last term 31). A1: 175

Alternative: (Listing all sums)

M1: Listing 9, 10 or 11 sums. (At least 4, 7, ….., "*last*").

A1: Correct sums, correct finishing value 175. A1: 175

Alternative: (Using last term).

M1: Using $S_n = \frac{n}{2}(a + l)$ with T_9, T_{10} or T_{11} as the last term.

A1: Correct numerical version $\frac{10}{2}(4 + 31)$. A1: 175

Correct answer with no working scores 1 mark: 1,0,0.

(c) $S_k < 1750$: $\frac{k}{2}\{8+3(k-1)\} < 1750 \left(\text{or } S_{k+1} > 1750 : \frac{k+1}{2}\{8+3k\} > 1750\right)$ M1

$3k^2 + 5k - 3500 < 0$ (or $3k^2 + 11k - 3492 > 0$) M1A1

(Allow equivalent 3-term versions such as $3k^2 + 5k = 3500$).

$(3k - 100)(k + 35) < 0$

Requires use of correct inequality throughout.(*) A1cso 4

For the first 3 marks, allow any inequality sign, or equals.

1st M: Use of correct sum formula to form inequality or equation in k, with the 1750.

2nd M:(Dependent on 1st M). Form 3–term quadratic in k.

1st A: Correct 3 terms.

Allow credit for part (c) if valid work is seen in part (d).

(d) $\frac{100}{3}$ or equiv. seen $\left(\text{or } \frac{97}{3}\right)$, $k = 33$ (and no other values) M1, A1 2

Allow both marks for $k = 33$ seen without working.

Working for part (d) must be seen in part (d), not part (c).

[12]

9 $a + (n - 1)d = k$ $k = 9$ or 11 M1

$(u_{11} =)\ a + 10d = 9$ A1c.a.o.

$\frac{n}{2}[2a+(n-1)d] = 77$ or $\frac{(a+1)}{2} \times n = 77$ $l = 9$ or 11 M1

$(S_{11} =)\ \frac{11}{2}(2a+10d) = 77$ or $\frac{(a+9)}{2} \times 11 = 77$ A1

eg $a + 10d = 9$ or $a + 9 = 14$
$a + 5d = 7$ M1

$a = 5$ and $d = 0.4$ or exact equivalent A1A1

1st M1 Use of u_n to form a linear equation in a and d.
$a + nd = 9$ is M0A0

1st A1 For $a + 10d = 9$.

2nd M1 Use of S_n to form an equation for a and d (LHS) or in a (RHS)

2nd A1 A correct equation based on S_n.

For 1st 2 Ms they must write n or use $n = 11$.

3rd M1 Solving (LHS simultaneously) or (RHS a linear equation in a)
Must lead to a = ... or d = ... and depends on one previous M

3rd A1 for $a = 5$

4th A1 for $d = 0.4$ (o.e.)

<u>ALT</u> Uses $\frac{(a+l)}{2} \times n = 77$ to get $a = 5$, gets second and third M1 A1 i.e. 4/7
Then uses $\frac{n}{2}[2a+(n-1)d]=77$ to get d, gets 1st M1 A1 and 4th A1

<u>MR</u> Consistent MR of 11 for 9 leading to $a = 3$, $d = 0.8$ scores M1A0M1A0M1 A1ftA1ft

[7]

10.	(a)	$a + (a + d) =$ £ (500 + 500 + 200) = £1200	cso	B1	1
	(b)	$a = 500$, $d = 200$; $u_8 = a + (8 – 1)d$		M1	
		= £(500 + 7 × 200) = £1900		A1	2
	(c)	$S_8 = \frac{8}{2}(2 \times 500 + (8 – 1) \times 200)$		M1 A1	
		= £ 9600		A1	3
	(d)	$\frac{n}{2}(1000 + (n – 1)\,200) = 32\,000$		M1 A1	
		$n^2 + 4n – 320 = 0$ M1 reducing to a 3 term quadratic A1 any multiple of the above		M1 A1	
		$(n + 20)(n – 16) = 0$		M1	
		$n = 16$		A1	
		Age is 26		A1	7

[13]

In (b) if the sum is found by repeated addition, i.e.

u_1 = £500, u_2 = £700, u_3 = £900, u_4 = £1100, u_5 = £1300,

u_6 = £1500, u_7 = £1700, u_8 = £1900,
allow M1 A1 at completion.

If for (c) these 8 terms are added up, allow M1 A2 at completion. Do notdivide the As with this method if (b) has been completed similarly. If only(c) is done by repeated addition allow A1 if the individual terms are correct if a complete method is shown.

11.	(a)	$S = a + (a + d) + (a + 2d) + \ldots + [a + (n – 1)d]$	B1	
		$S = [a + (n – 1)d] + [a + (n – 2)d] + \ldots + a$ or equiv.	M1	
		Add: $2S = n[2a + (n – 1)d] \Rightarrow S = n[2a + (n – 1)d]$ cso (*)	M1 A1	4
		B1: requires min of 3 terms, including the last.		
	(b)	3, 8, 13	B1	1
		First M1 generous; second M1 hard.		
		Note: Result is given so check working carefully.		

(c)	$a = 3 \quad d = 5$	B1ft	
	Sum = $\frac{1}{2}n[(2 \times 3) + 5(n - 1)] = \frac{1}{2}n(5n + 1)$ (*)	M1 A1	3
	For B1 f.t. 3 terms must be in AP, *But allow M1 for candidate's "a" and "d" in given result in (a)*		
	EXTRA		
	$5\sum r - \sum 2$	B1	
	$= 5\frac{n(n+1)}{2} - 2n$	M1	
	$= \frac{5n^2 + n}{2} = \frac{n(5n+1)}{2}$ (*) (cso)	A1	
(d)	Finding $\sum_{1}^{200}$ e.g. $\sum_{r=1}^{200}(5r-2) = \frac{1}{2} \times 200 \times 1001$ (= 100100)	M1	
	Sum of first 4 terms: $\sum_{r=1}^{4}(5r-2) = \frac{1}{2} \times 4 \times 21$ or 42 stated	B1	
	$\sum_{r=5}^{200}(5r-2)$ = S(200) – S(4) = 100100 – 42 = 100058	M1A1	4
	ALT: Working with 23, 28, 33, ………… a = 23 B1; Finding "n" and d M1 Applying $S = \frac{1}{2}n[2a + (n-1)d]$ with candidate's 23, n= 195 or 196, $d = 5$	M1	
	First M1 for substitution of 200 in result from (c) *S.C. Allow second M1 for S(200) – S(5)*		

[12]

12. (a)	$(S =)a + (a + d) + \dots \quad \dots + [a + (n-1)d]$	B1		
	$(S =)[a + (n-1)d] + \dots \quad \dots + a$	M1		
	$2S = [2a + (n-1)d] + \dots \quad \dots + [2a + (n-1)d]$ } either	dM1		
	$2S = n[2a + (n-1)d]$			
	$S = \frac{n}{2}[2a + (n-1)d]$	A1 c.s.o.	4	

B1 requires at least 3 terms, must include first and last terms, an adjacent term and dots + signs.

1st M1 for reversing series. Must be arithmetic with a, n and d or l. (+ signs not essential here)

2nd dM1 for adding, must have 2S and be a genuine attempt. Either line is sufficient.

Dependent on 1st M1

(NB Allow first 3 marks for use of l for last term but as given for final mark)

(b)	$(a = 149,\ d = -2)$		
	$u_{21} = 149 + 20(-2) = £109$	M1 A1	2

M1 for using a = 149 and d = ±2 in a + (n – 1)d formula.

(c)	$S_n = \frac{n}{2}[2 \times 149 + (n-1)(-2)] \quad (n(150 - n)$	M1 A1	
	$S_n = 5000 \Rightarrow n^2 - 150n + 5000 = 0$ (*)	A1 c.s.o.	3

M1 for using their a, d in S_n A1 any correct expression

A1cso for putting S_n = 5000 and simplifying to given expression. No wrong work

(d)	$(n - 100)(n - 50) = 0$	M1	
	$n = 50$ or 100	A2/1/0	3

M1 Attempt to solve leading to n = …
A2/1/0 Give A1A0 for 1 correct value and A1A1 for both correct

(e)	$u_{100} < 0 \ \therefore n = 100$ not sensible	B1 ft	1

B1 f.t. Must mention 100 and state u_{100} < 0 (or loan paid or equivalent)

If giving f.t. then must have n ≥ 76.

[13]

13.	(a)	$-3, \; -1, \; 1$	B1 B1	2
		B1: One correct		
	(b)	2	B1ft	1
		(ft only if terms in (a) are in arithmetic progression)		
	(c)	Sum = $\frac{1}{2}n\{2(-3)+(n-1)(2)\}$ or $\frac{1}{2}n\{(-3)+(2n-5)\}$	M1 A1ft	
		$= \frac{1}{2}n\{2n-8\} = n(n-4)$ (<u>Not</u> just $n^2 - 4n$) (*)	A1	3
				[6]

14.	(a)	$(5p - 8) - p = (3p + 8) - (5p - 8)$	M1	
		Solve, showing steps, to get $p = 4$, or verify that $p = 4$. (*)	A1 c.s.o.	2
		<u>Alternative:</u> Using $p = 4$, finding terms (4, 12, 20), and indicating differences.	[M1]	
		Equal differences + conclusion (or "common difference = 8").	[A1]	
	(b)	$a = 4$ and $d = 8$ (stated or implied here or elsewhere).	B1	
		$T_{40} = a + (n - 1)d = 4 + (39 \times 8) = 316$	M1 A1	3
	(c)	$S_n = \frac{1}{2}n[2a + (n - 1)d] = \frac{1}{2}n[8 + 8(n - 1)]$	M1 A1ft	
		$= 4n^2 = (2n)^2$	A1	3
				[8]

15.	(a)	77 74	B1 B1	2
	(b)	$d = 74 - 77 = -3$	B1 ft	1
	(c)	$S_{50} = \frac{1}{2}n[2a + 9n - 1)d] = 25[(2 \times 77) + (49 \times -3)]$	M1 A1 ft	
		$= 175$	A1	3
		Alternative method: Find last term, then use $\frac{1}{2}n(a + 1)$		
				[6]

16.	(a)	(i) $a + (n-1)d = 280 + (35 \times 5) = 455$	M1 A1	2	
		(ii) $\frac{1}{2}n\,[2a + (n-1)d\,] = 18\,[560 + (35 \times 5)] = 13\,230$	M1 A1 ft	2	
	(b)	$18\,[560 + (35 \times d)] = 17\,000$	M1 A1		
		$d = 10.98\ldots$ $x = 11$ (allow 11.0 or 10.98 or 10.99 or $10\frac{62}{32}$)	M1 A1	4	
					[8]
17.	(a)	$S = a + (a+d) + (a+2d) + \ldots + a + (n-1)d$	B1		
		$S = a + (n-1)d + a + (n-2)d + \ldots + a$	M1		
		Adding,			
		$2S = n[2a + (n-1)d] \Rightarrow S = \frac{n}{2}[2a + (n-1)d]$	M1 A1	4	
	(b)	$k + k + 7 = 2 \times 7.5 \Rightarrow k = 4$	M1 A1	2	
	(c)	$S_{31} = \frac{31}{2}[2\,(4) + 30\,(3\frac{1}{2})]$	M1 A1 ft		
		1751.5	M1 A1	4	
					[10]
18.	(a)	$S = a + (a+d) + \ldots.. + [a + (n-1)d]$	B1		
		$S = [a + (n-1)\,d] + \ldots\ldots\ldots\ldots\ldots + a$	M1		
		Add: $2S = n[2a + (n-1)d]$, $S = \frac{1}{2}n[2a + (n-1)d]$ **(*)**	M1 A1	4	
	(b)	$a + 15d = 6$	B1		
		$\frac{1}{2}n[2a + (n-1)d] = 8\,(2a + 15d) = 72$	M1 A1		
		Solve simultaneously: $a = 3$ 3cm	M1 A1	5	
	(c)	$a = 3$: $15d = 6 - 3 = 3$ $d = 0.2$	M1 A1	2	
					[11]

Year 2: A Level Mathematics	
<u>Statistics: Sampling</u>	
Self-Assessment: Please identify areas in which you believe are your strong points and those you feel you need to improve on Provide evidence to support your assessment with reference to the content in this booklet.	
Strengths	**Areas for Improvement**

Data Collection Cheat Sheet

Population and sample

In statistics, population is the whole set of items that are of interest. Information obtained from a population is known as raw data. A census measures or observes every member of a population. A sample is a selection of observations taken from a subset of population and used to find out more information about the population as a whole.

	Advantages	Disadvantages
Census	• Results should be completely accurate	• Time consuming and expensive • Cannot be used when testing destroys process • Hard to process large quantity of data
Sample	• Less time consuming and cheaper • Fewer people have to respond • Less data needs to be processed	• Data may not be as accurate • Sample may not be large enough to give information about small subgroups of the population

Individual units of a population are known as sampling units. Sampling units are named and numbered to form a list called a sampling frame.

Random sampling

Each member of the population has an equal chance of being selected. The sample should be representative of the population and bias should be removed. There are 3 types of random sampling.

- Simple random sampling

A simple random sample of size n is one where every sample of size n has an equal chance of being selected.

Example 1: The 100 members of a yacht club are listed alphabetically in the club's membership book. The committee wants to select a sample of 12 members to fill in a questionnaire. Explain how a simple random sample can be taken using:

A) Calculator or random number generator:
Number each member from 1-100. Use a calculator or random number generator to generate 12 random numbers between 1-100. Select the members who correspond to the numbers.

B) Lottery sampling:
Write the name of members on identical cards and place them in the hat. Draw up 12 cards and select these members.

Advantages	Disadvantages
• Free of bias • Easy and cheap for small samples and populations • Each sampling unit has a known and equal chance of selection	• Not suitable for large samples and populations • Sampling frame needed

- Systematic sampling

The required elements are chosen at regular intervals from an ordered list.

Example 2: A sample of size 20 is required from a population of 100.

$100 \div 20 = 5$ so every fifth person is chosen.
The first person is chosen at random.
If the first person chosen is 2, the remaining samples will be 7, 12, 17 etc.

Advantages	Disadvantages
• Simple and quick to use • Suitable for large samples and large populations	• A sampling frame is needed • Bias introduced if sampling frame is not random

- Stratified sampling

The population is divided into mutually exclusive strata and a random sample is taken from each.

Number sampled in a stratum$=\frac{\text{number in stratum}}{\text{number in population}}$ x overall sample size

Example 3: A factory manager wants to find out about what his workers think about the factory canteen facilities. He decides to give a questionnaire to a sample of 80 workers. It is thought that different age groups will have different opinions.

There are 75 workers between ages 18 and 32, 140 workers between ages 33 and 47, and 85 workers between ages 48 and 62.

Explain how he can use stratified sampling to select the sample.

1. Total number of workers: $75 + 140 + 85 = 300$
2. Finding the number of workers needed from each age group:
   18-32: $\frac{75}{300} \times 80 = 20$ workers
   33-47: $\frac{140}{300} \times 80 = 37\frac{1}{3} \approx 37$ workers
   48-62: $\frac{85}{300} \times 80 = 22\frac{2}{3} \approx 23$ workers
   If the number of workers required is not a whole number, it is rounded off to the nearest whole number.
3. Number the workers in each group.
4. Use a random number generator or table to produce the required quantity of random numbers.

Advantages	Disadvantages
• Sample accurately reflects population structure • Proportional representation of group within population	• Population must be clearly classified into distinct strata • Same disadvantages as simple random sampling within each stratum

Non-random sampling

There are two types of non-random sampling that you need to know:

- Quota sampling

An interviewer or researcher selects a sample that reflects the characteristics of the whole population.

Advantages	Disadvantages
• Allows a small sample to still be representative of the population • No sampling frame required • Quick, easy and inexpensive • Easy comparison between different groups within a population	• Non-random sampling can introduce bias • Population must be divided into groups, which can be costly or inaccurate • Increasing scope of study increases number of groups, which adds time and expenses • Non-responses not recorded

- Opportunity sampling or convenience sampling

Sample is taken from people who are available at the time of study and who fits the criteria you are looking for.

Advantages	Disadvantages
• Easy and inexpensive	• Unlikely to provide a representative result • Highly dependent on individual researcher

Types of data

Variables or data associated with numerical observations are called quantitative variables or quantitative data.

Variables associated with non-numerical observations are qualitative variables or qualitative data.

A variable that can take any value in a given range is a continuous variable. A variable that can only take specific values is a discrete variable.

In a grouped frequency table, the specific data values are not shown.

- Class boundaries show the maximum and minimum values in each group or class
- The midpoint is the average of class boundaries
- The class width is the difference between upper and lower class boundaries

Large data set

If you need to do calculations on large data sets in your exam, the relevant extract will be provided.

Measures of Location and Spread Cheat Sheet

Edexcel Stats/Mech Year 1

Measures of central tendency

A measure of central tendency describes the centre of the data. You need to decide of the best measure to use in particular situations.

The mode or modal class is the value of class which occurs most often. This is used when data is qualitative or quantitative with one mode or two modes (bimodal). It is not informative if each value only occurs once.

The median is the middle value when the data values are put in order. This is used for quantitative data and usually used when there are extreme values as they are unaffected.

The mean can be calculated using:

$$\bar{x} = \frac{\Sigma x}{n}$$

Where $\bar{x}$ (x bar) is the mean,
Σx is the sum of the data values,
n is the number of data values

For data given in a cumulative frequency table, the mean can be calculated using:

$$\bar{x} = \frac{\Sigma xf}{\Sigma f}$$

Where Σfx is the sum of the products of the data values and their frequencies,
Σf is the sum of frequencies

The mean is used for quantitative data. It uses all values in the data therefore it gives a true measure of data. However, it is affected by extreme values.

You can calculate the mean, class containing median and modal class for continuous data presented in a grouped frequency table by finding the midpoint of each class interval.

Other measures of location

The median(Q_2) splits the data into two equal halves (50%).
The lower quartile (Q_1) is one quarter of the way through the dataset.
The upper quartile (Q_3) is three quarters of the way through the dataset.

Percentiles split the data set into 100 parts. The 10th percentile is one-tenth of the way through the data, for example. 10% of data values are less than the 10th percentile and 90% are greater.

To find lower and upper quartiles for discrete data:

1. Divide n by 4. (lower quartile) OR
   Find $\frac{3}{4}$ of n. (upper quartile)
2. If this is a whole number, the lower or upper quartile is the midpoint between this data point and the number above. If it is not, round up and pick this number.

When data is presented in a grouped frequency table, you can use interpolation to estimate the medians, quartiles, and percentiles. This method assumes that the data values are evenly distributed within each class.

$Q_1 = \frac{n}{4}$th data value
$Q_2 = \frac{n}{2}$th data value
$Q_3 = \frac{3n}{4}$th data value

Measures of spread

Measures of spread shows how spread out the data is. They are also known as measures of dispersion or measures of variation.

- Range
The difference between largest and smallest values in the dataset.
- Interquartile range (IQR)
The difference between upper and lower quartile.
- Interpercentile range
Difference between the values of two given percentiles.

Variance (σ^2) and standard deviation (σ)

The variance also shows how spread out the data is. There are 3 versions of the formulae used to find variance:

1. $\sigma^2 = \frac{\Sigma(x-\bar{x})^2}{n}$
2. $\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

Easier to use when given raw data

You can remember this as “mean of squares minus square of means”

3. $\sigma^2 = \frac{S_{xx}}{n}$

Where $S_{xx} = \Sigma(x-\bar{x})^2 = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$

You can use calculator to calculate S_{xx}

S_{xx} is a summary statistic used to simplify formula

Standard deviation is the square root of variance.

$$\sigma = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2} = \sqrt{\frac{S_{xx}}{n}}$$

For grouped data presented in frequency table:

$$\sigma^2 = \frac{\Sigma f(x-\bar{x})^2}{\Sigma f} = \frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2$$

$$\sigma = \sqrt{\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}} = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2}$$

Where f is the frequency of each group and Σf is the total frequency

Example 1: Shamsa records the time spent out of school during lunch hour to the nearest minute, x, of the students in her year in the table below. Calculate the standard deviation.

Time spent out of school (min)	35	36	37	38
Frequency	3	17	29	34

1. Find Σfx^2, Σfx and Σf
   $\Sigma fx^2 = 3 \times 35^2 + 17 \times 36^2 + 29 \times 37^2 + 34 \times 38^2$
   $= 114504$
   $\Sigma fx = 3 \times 35 + 17 \times 36 + 29 \times 37 + 34 \times 38$
   $= 3082$
   $\Sigma f = 3 + 17 + 29 + 34 = 83$
2. Use formula for grouped data in frequency table to find variance:
   $\sigma^2 = \frac{114504}{83} - \left(\frac{3082}{83}\right)^2 = 0.74147 \ldots$
3. Square root variance to find standard deviation:
   $\sigma = \sqrt{0.74147 \ldots} = 0.861$ (3s.f.)

Coding

Each value in the data can be coded to give a new set of values, which is easier to work with. Coding also changes different statistics in different ways.

If data is coded using the formula $y = \frac{x-a}{b}$, where a and b are constants that you have to choose or given in the question:

- Mean of coded data: $\bar{y} = \frac{\bar{x}-a}{b}$
  Rearrange the formula to find original mean: $\bar{x} = b\bar{y} + a$
- Standard deviation of coded data: $\sigma_y = \frac{\sigma_x}{b}$
  Rearrange the formula to find original standard deviation: $\sigma_x = b\sigma_y$

Example 2: A scientist measures the temperature, x°C, at five different points of a nuclear reactor. Her results are given below:
332°C, 355°C, 306°C, 317°C, 340°C

a. Use the coding $y = \frac{x-300}{10}$ to code this data.

Substitute each value into x to get coded data, y.

Original data, x	332	355	306	317	340
Coded data, y	3.2	5.5	0.6	1.7	4.0

b. Calculate the mean and standard deviation of the coded data.
   $\Sigma y = 15$, $\Sigma y^2 = 59.74$
   $\bar{y} = \frac{15}{5} = 3$
   $\sigma_y^2 = \frac{59.74}{5} - \left(\frac{15}{5}\right)^2 = 2.948$
   $\sigma_y = \sqrt{2.948} = 1.72$ (3s.f.)

c. Calculate the mean and variance of the original data using your answers from part b.
   $3 = \frac{\bar{x}-300}{10}$ so $\bar{x} = 330$°C
   $1.72 = \frac{\sigma_x}{10}$ so $\sigma_x = 17.2$°C (3s.f.)

Representations of Data Cheat Sheet

Edexcel Stats/Mech Year 1

Outliers

An outlier is commonly any value which fits into one of the following:

- Greater than $Q_3 + k(Q_3 - Q_1)$
- Less than $Q_1 - k(Q_3 - Q_1)$

The value of k will be given in the exam

Some questions have other ways of identifying the outliers. In the exam, you will be told which method to use.

Example 1: Some data is collected. $Q_1 = 46$ and $Q_3 = 68$. A value greater than $Q_3 + k(Q_3 - Q_1)$ or less than $Q_1 - k(Q_3 - Q_1)$ is defined as an outlier. Work out if a)7, b)88 and c)105 are outliers. The value of k is 1.5.

$$68 + 1.5(68 - 46) = 101$$
$$46 - 1.5(68 - 46) = 13$$

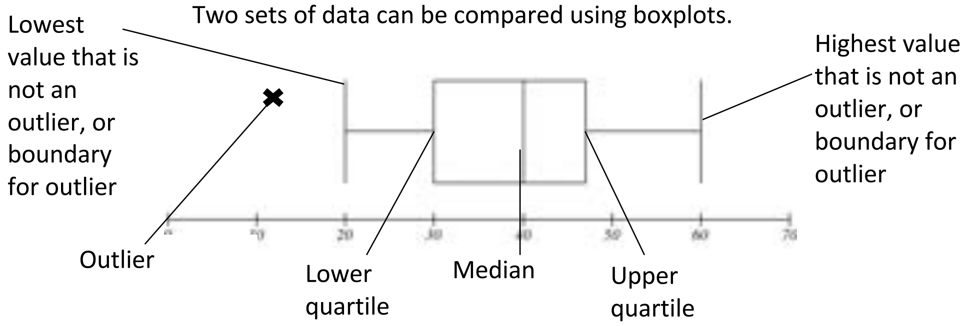
7<13 and 105>101 so 7 and 105 are outliers, 88 is not an outlier.

In some cases, the outliers are legitimate values which will still be correct. Some outliers are clearly an error and they are called anomalies. They can be due to experimental or recording error, or data values not relevant to the study. The process of removing anomalies from a data set is called data cleaning.

Boxplots

A boxplot shows the quartiles, maximum and minimum values and any outliers

Two sets of data can be compared using boxplots.

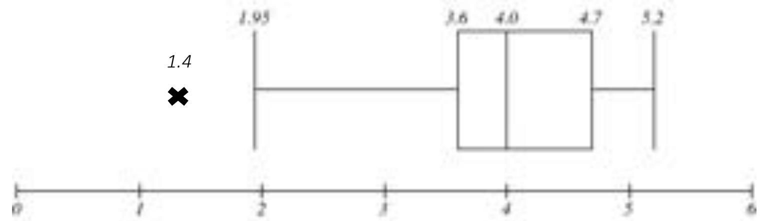


Example 2: The blood glucose level of 30 males is recorded. The results, in mmol/litre, are summarised below:

Lower quartile: 3.6
Upper quartile: 4.7
Median: 4.0
Lowest value: 1.4
Highest value: 5.2

An outlier is an observation that falls either 1.5x interquartile range above the upper quartile or 1.5x interquartile range below the lower quartile.

a. Given that there is only one outlier, draw a boxplot for this data:
   1. Calculate the value of outlier:
      $3.6 - 1.5 \times 1.1 = 1.95$
      $4.7 + 1.5 \times 1.1 = 6.35$
      1.4 < 1.95, therefore the outlier is 1.4.
   2. Draw the boxplot and label the axis.
      The end of the whisker is plotted at the outlier boundary since the actual figure is not known.



Blood glucose level (mmol/litre)

Cumulative Frequency

You can use a cumulative frequency diagram to help find estimates for the median, quartiles and percentiles in a grouped frequency table.

Histograms

Group continuous data can be presented using histograms. Histograms show the rough location and general shape of the data, and how spread out the data is.

The area of the bar is proportional to the frequency of each class.

To calculate the height of each bar (frequency density):

Area of bar= $k \times$ frequency

When $k = 1$,

$$\text{Frequency density} = \frac{\text{frequency}}{\text{class width}}$$

Joining the middle of the top of each bar in histogram forms a frequency polygon.

Example 3: A random sample of 200 students was asked how long it took them to complete their homework. Their responses are summarised in a table:

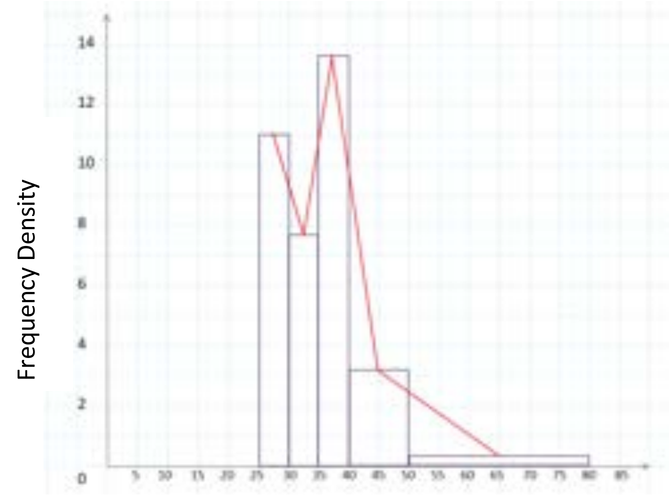
Time, t(min)	$25 \le t < 30$	$30 \le t < 35$	$35 \le t < 40$	$40 \le t < 50$	$50 \le t < 80$
Frequency	55	39	68	32	6

a. Draw a histogram and frequency polygon to present the data.
   1. Find the class width and frequency density of each class.

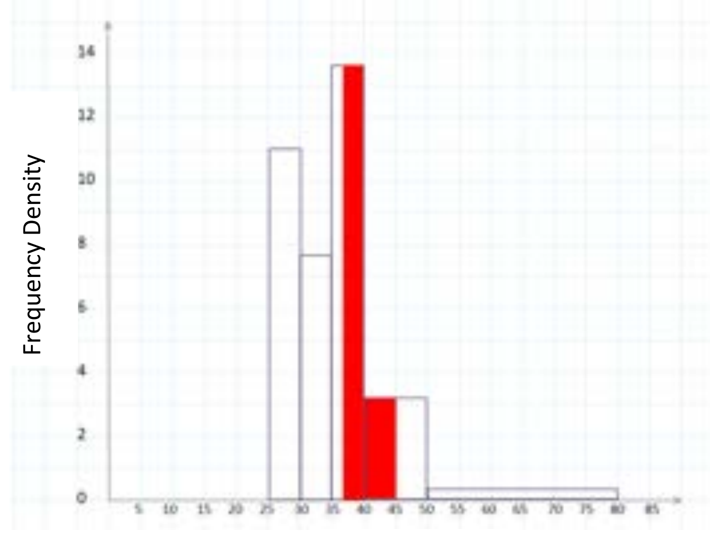
Time, t(min)	Frequency	Class width	Frequency density
$25 \le t < 30$	55	5	11
$30 \le t < 35$	39	5	7.8
$35 \le t < 40$	68	5	13.6
$40 \le t < 50$	32	10	3.2
$50 \le t < 80$	6	30	0.2

$$\text{Frequency density} = \frac{\text{frequency}}{\text{class width}}$$

   2. Draw the histogram using class width as the width of each bar and frequency density as the height.
   3. To draw the frequency polygon, join the middle of the top of each bar of the histogram.



b. Estimate how many students took between 36 and 45 minutes to complete their homework.



The number of students is directly proportional to the area under graph between 36 and 45 minutes.

Area: $(40 - 36) \times 13.6 + (45 - 40) \times 3.2 = 70.4$ students

Comparing data

When comparing data, you can comment on

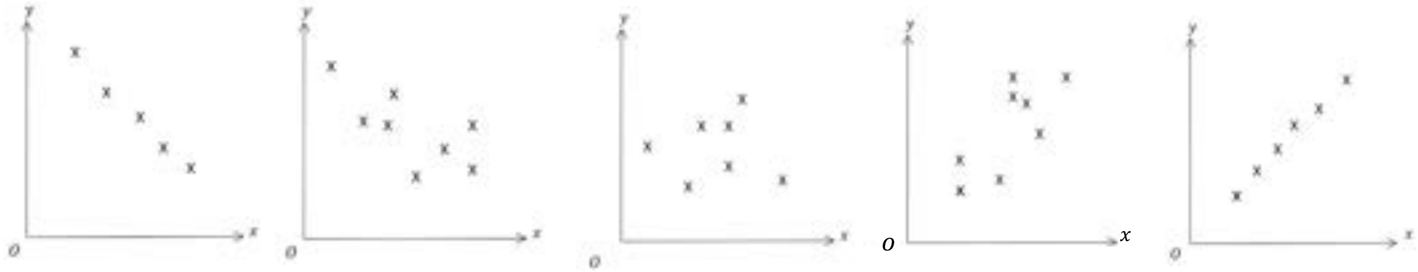
- A measure of location
- A measure of spread

You can use the mean and standard deviation or median and interquartile range (suitable for data sets with extreme values)
Median should not be used with standard deviation and mean should not be used with interquartile range.

Correlation Cheat Sheet

Bivariate data is data which has pairs of values for two variables. You can represent bivariate data on a scatter diagram. The independent or explanatory variable is something which the researcher can control and is usually plotted on the x-axis. The dependent or response variable, which is measured by the researcher, is usually plotted on the y-axis.

Correlation describes the nature of linear relationships between two variables.



Strong negative correlation | Weak negative correlation | No linear correlation | Weak positive correlation | Strong positive correlation

A negative correlation means that one variable decreases when the other increases. Positive correlation means that one variable increases with the increase of the other variable.

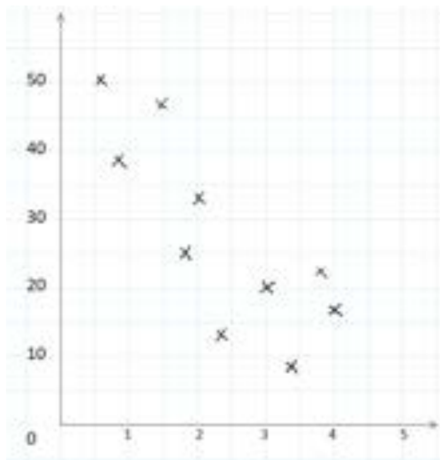
Two variables have a causal relationship if a change in one variable causes a change in the other. If two variables are correlated, you need to look at the context of the question to determine if they have a causal relationship.

Example 1: In the study of a city, the population density, in people/hectare, and the distance from the city centre, in km, was investigated by picking a number of sample areas with the following results.

Area	A	B	C	D	E	F	G	H	I	J
Distance (km)	0.6	3.8	2.4	3.0	2.0	1.5	1.8	3.4	4.0	0.9
Population density (people/hectare)	50	22	14	20	33	47	25	8	16	38

a. Draw a scatter diagram to represent this data.

Population density (people/hectare)



Distance from centre (km)

Remember to label your axis and include units

b. Describe the correlation between distance and population density.

There is a weak negative correlation.

Describe the strength of correlation and whether it is positive or negative

c. Interpret your answer to part b.

As distance from the centre increases, the population density decreases.

Interpret results in context to the question

Linear regression

The least squares regression line, or regression line, is a line of best fit which can be drawn on a scatter plot. This is the straight line that minimises the sum of the squares of the distances of each datapoint from the line.

The regression line of y on x is written in the form:

$$y = a + bx$$

The coefficient b tells you the change in y for each unit change in x.

- For positively correlated data, b is positive
- For negatively correlated data, b is negative

You can substitute a known value of the independent variable into x and use the regression line to estimate the corresponding value of the dependent variable. This should only be done within the range of data given and is known as interpolation. Extrapolation out of the data range gives a much less reliable estimate.

If you need to predict a value of x for a given value of y, you will need to use the regression line of x on y.

Example 2: The daily mean windspeed, w knots, and the daily maximum gust, g knots, were recorded for the first 15 days in May in Camborne. The data was plotted on a scattered diagram. The equation of the regression line of g on w for these 15 days is $g = 7.23 + 1.82w$.

a. Give an interpretation of the value of the gradient of this regression line.
The daily maximum gust is expected to increase by approximately 1.8 knots when the daily mean windspeed increases by 1 knot.

b. Predict the daily maximum gust when the daily mean speed is 16 knots.
Substitute 16 into w in the regression equation:
$g = 7.23 + 1.82(16)$
$= 36.35$ knots

Probability Cheat Sheet

Edexcel Stats/Mech Year 1

Calculating probabilities

An experiment is a repeatable process that gives rise to a number of outcomes. An event is a collection of one or more outcomes. A sample space is the set of all possible outcomes.

Probabilities can be written as decimals or fractions and are in the range of 0(impossible) to 1(certain).

If each outcome has an equal likelihood of occurring,

$$\text{Probability of event} = \frac{\text{number of possible outcomes in the event}}{\text{total number of possible outcomes}}$$

Example 1: The table shows the time taken, in minutes, for a group of students to complete a number puzzle.

Time, t(min)	$5 \le t < 7$	$7 \le t < 9$	$9 \le t < 11$	$11 \le t < 13$	$13 \le t < 15$
Frequency	6	13	12	15	4

A student is chosen at random. Find the probability that they finished the number puzzle:

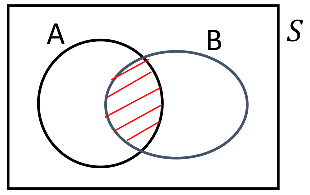
a. In under 9 minutes
Total number of students: $6 + 13 + 12 + 15 + 4 = 50$
Number of students who finished under 9 minutes: $6 + 13 = 19$
P (finished under 9 minutes) = $\frac{19}{50}$

b. In over 10.5 minutes
10.5 minutes is $\frac{3}{4}$ through the $9 \le t < 11$ class. Estimate using interpolation:
$\frac{1}{4} \times 12 = 3$
$3 + 15 + 4 = 22$
P (finished in over 10.5 minutes) = $\frac{22}{50}$

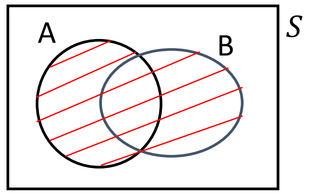
Venn Diagrams

A Venn diagram can be used to represent events graphically. Frequencies or probabilities can be placed in the regions of Venn diagrams.

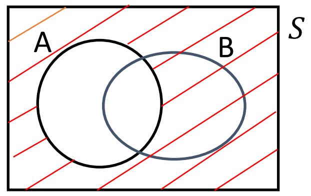
A rectangle represents the sample space, S. It contains closed curves which represent events.



Intersection of A and B shows the event in which both A and B occur



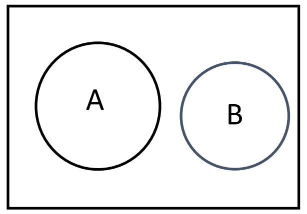
Union of A and B shows the event in which either A or B or both occur



The shaded area shows the event in which A does not occur

Mutually exclusive independent events

Events which have no outcomes in common are called mutually exclusive. The closed curves do not overlap in a Venn Diagram.



For mutually exclusive events,

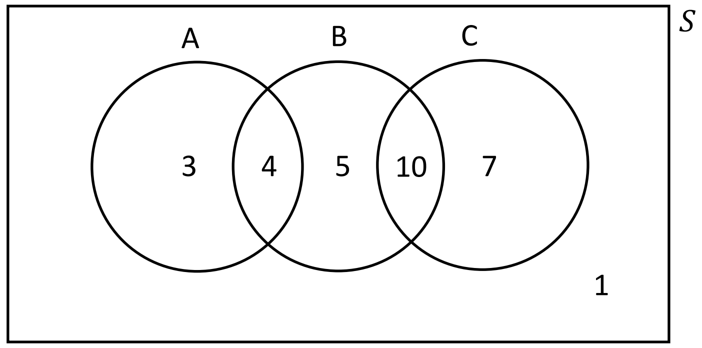
P(A or B) =P(A)+P(B)

When one event has no effect on another, they are independent. For independent events A and B, the probability of B happening is the same regardless of whether A happens.
For independent events,

P(A and B)=P(A)×P(B)

You can also use this multiplication rule to check if events are independent.

Example 2: The Venn diagram shows the number of students in a particular class who watch any of three popular TV shows



a. Find the probability of a student chosen at random watches B or C or both.
$4 + 5 + 10 + 7 = 26$
P(watches B or C or both)= $\frac{26}{30} = \frac{13}{15}$

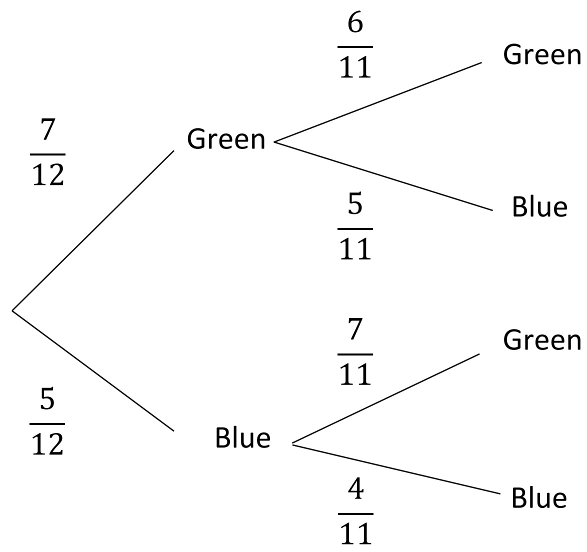
b. Determine whether watching A and watching B are statistically independent.
P(A)= $\frac{3+4}{30} = \frac{7}{30}$
P(B)= $\frac{4+5+10}{30} = \frac{19}{30}$
P(A and B)= $\frac{4}{30} = \frac{2}{15}$
P(A)×P(B)= $\frac{7}{30} \times \frac{19}{30} = \frac{133}{900}$
P(A and B)≠ P(A)×P(B)
Therefore, watching A and watching B are not statistically independent.

Tree diagrams

A tree diagram can be used to show the outcomes of two or more events happening in succession.

Example: A bag contains seven green beads and five blue beads. A bead is taken from the bag at random and not replaced. A second bead is then taken from the bag. Find the probability that:

a. Both beads are green
1. Draw a tree diagram to show the events.



2. Multiply along the branch of tree diagram:
P(green and green) $= \frac{7}{12} \times \frac{6}{11} = \frac{7}{22}$

b. The beads are different colours
P(different colours) = P(green then blue) + P(blue then green)
$= \frac{7}{12} \times \frac{5}{11} + \frac{5}{12} \times \frac{7}{11}$
$= \frac{35}{66}$

Statistical Distributions Cheat Sheet

Probability distributions

A variable can take any of a range of specific values. A variable is random if the outcome is not known until the experiment is carried out. Random variables are written as upper case letters, for example X or Y. The particular values the random variable can take are written as equivalent lower case letters, for example x or y. The probability that the random variable X takes a particular value x is written as:

$$\mathrm{P}(X = x)$$

A probability distribution fully describes the probability of any outcome in the sample space. The probability distribution of a discrete random variable can be describe using probability mass function, a table or a diagram.

When all probabilities are the same, the distribution is known as discrete uniform distribution. For example, the score when a fair dice is rolled.

Example 1: Three fair coins are tossed. A random variable, X is defined as the number of heads when the three coins are tossed. Shown the probability distribution as X as a a) table, b) probability mass function, c) diagram.

All possible outcomes when the coins are tossed:
HHH, HHT, HTH, THH, HTT, THT, TTH, TTT
Since the coins are fair, the probability of getting each outcome listed above is the same.

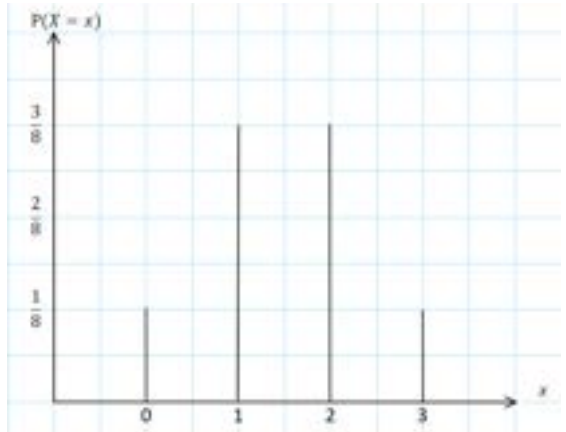
a. Table

No. of heads, x	0	1	2	3
$\mathrm{P}(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b. Probability mass function:

$$\mathrm{P}(X = x)\begin{cases} \frac{1}{8} & x = 0,3 \\ \frac{3}{8} & x = 1,2 \\ 0 & \text{otherwise} \end{cases}$$

c. Diagram:



Binomial distribution

You can define a random variable X to represent the number of successful trials when you are carrying out a number of trials.

You can model X with a binomial distribution $\mathrm{B}(n, p)$ if:

- There are a fixed number of trials, n
- There are two possible outcomes (success and failure)
- There is a fixed probability of success, p
- The trials are independent of each other

If a random variable X has the binomial function $\mathrm{B}(n, p)$, then its probability mass function is given by:

$$\mathrm{P}(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$$

$$\boxed{\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = {}^{n}\mathrm{C}_{r}}$$

n is also called the index and p is called the parameter.

You can also use the binomial probability distribution function in the calculator to work out the binomial probabilities.

Example 2: The probability that a randomly chosen member of a reading group is left-handed is 0.15. A random sample of 20 members of the group is taken.

a. Suggest a suitable model for the random variable X, the number of members in the sample who are left-handed. Justify your choice.

The random variable can only take two values, left-handed or right-handed. There are a fixed number of trials: 20, and a fixed probability of success: 0.15. Assuming each member in the sample is independent, a suitable model is $X \sim \mathrm{B}(20, 0.15)$.

b. Use your model to calculate the probability that:

i. Exactly 7 of the members in the sample are left-handed

$$\mathrm{P}(X = 7) = \binom{20}{7} \times (0.15)^7 (0.85)^{13}$$
$$= 0.01601 \ldots$$
$$= 0.0160 \text{ (3 s.f.)}$$

ii. Fewer than two of the members in the sample are left-handed

$$\mathrm{P}(X < 2) = \mathrm{P}(X = 0) + \mathrm{P}(X = 1)$$
$$= 0.03875 \ldots + 0.13679 \ldots$$
$$= 0.176 \text{ (3s.f.)}$$

Cumulative probabilities

$\mathrm{P}(X \le x)$ gives the sum of all individual probabilities for values up to and including x.
$\mathrm{P}(X < x)$ gives the sum of all individual probabilities for values not greater than x.
$\mathrm{P}(X \ge x)$ gives the sum of all individual probabilities for x and values greater than x.
$\mathrm{P}(X > x)$ gives the sum of all individual probabilities for values greater than x.

You can use the tables in the formula book or the binomial cumulative probability function in the calculator to find cumulative probabilities for $X \sim \mathrm{B}(n, p)$.

Example 3: A spinner is designed so that the probability it lands on red is 0.3. Jane has 12 spins. Find the probability that Jane obtains:

a. No more than 2 reds

Let X = number of reds in 12 spins.
$X \sim \mathrm{B}(12, 0.3)$
$\mathrm{P}(X \le 2) = 0.2528$

b. At least 5 reds

$$\mathrm{P}(X \ge 5) = 1 - \mathrm{P}(X \le 4)$$
$$= 1 - 0.7237$$
$$= 0.2763$$

Jane decides to use this spinner for a class competition. She wants the probability of winning a prize to be <0.05. Each student will have 12 spins and the number of reds will be recorded. Find out how many reds are needed to win a prize.

Let r =the smaller number of reds needed to win a prize
We need to find the value of r so that $\mathrm{P}(X \ge r) < 0.05$

From the table:
$\mathrm{P}(X \le 5) = 0.8822$
$\mathrm{P}(X \le 6) = 0.9614$
$\mathrm{P}(X \le 7) = 0.9905$

$x = 6$ is the first value which gives a probability greater than 0.95, so we use the probability of $\mathrm{P}(X \le 6)$ to find r.
$\mathrm{P}(X \le 6) = 0.9614$ implies that :
$$\mathrm{P}(X \ge 7) = 1 - 0.9614$$
$$= 0.0386 < 0.05$$

$\therefore r = 7$

7 or more reds are needed to win a prize.

Hypothesis Testing Cheat Sheet

Hypothesis testing

A hypothesis is a statement made about the value of a population parameter. It can be tested by carrying out an experiment or taking a sample from the population. The statistic calculated from the sample is called the test statistic.

The null hypothesis (H_0) is the hypothesis assumed to be correct. This is rejected if the test statistics is lower than a given threshold, called the significance level.

The alternative hypothesis (H_1) tells us about the parameter if your assumption is shown to be wrong.

Example 1: John wants to see if a coin is unbiased or biased towards coming down heads. He tosses the coin 8 times and counts the number of heads, X, obtained in 8 tosses.

a. Describe the test statistic.
The test statistic is X, the number of heads obtained in 8 tosses.

b. Write down a suitable null hypothesis.
The probability of landing heads for an unbiased coin is 0.5 so
$\mathrm{H}_0: p = 0.5$

c. Write down a suitable alternative hypothesis.
The probability for heads is greater than 0.5 if the coin is biased towards heads so:
$\mathrm{H}_1: p > 0.5$

Finding critical values

A critical region is a region of the probability distribution which, if the test statistic falls within it, would cause you to reject the null hypothesis. The critical value is the first value to fall inside of the critical region.

The actual significance level of a hypothesis test is the probability of incorrectly rejecting the null hypothesis.

Example 2: A single observation is taken from the binomial distribution $\mathrm{B}(6, p)$. The observation is used to test $\mathrm{H}_0: p = 0.35$ against $\mathrm{H}_1: p > 0.35$

a. Using a 5% significance level, find the critical region for this test.
Assume H_0 is true then $X \sim \mathrm{B}(6, 0.35)$
$\mathrm{P}(X \geq 4) = 1 - \mathrm{P}(X \leq 3)$
$= 1 - 0.8826$
$= 0.1174$
$\mathrm{P}(X \geq 5) = 1 - \mathrm{P}(X \leq 4)$
$= 1 - 0.9777$
$= 0.0223$
The critical region is 5 or 6.

You can use the cumulative binomial tables or your calculator

b. State the actual significance level of this test.
P(reject null hypothesis)$= \mathrm{P}(X \geq 5)$
$= 0.0223$
$= 2.23\%$

This is the same as the probability of X falling within the critical region

One-tailed test

A one-tailed test can be used to test if the probability has increased or decreased.

For one-tailed tests,

$$\mathrm{H}_1: p > \cdots \text{ or } p < \cdots$$

Example 3: The standard treatment for a particular disease has a $\frac{2}{5}$ probability of success. A researcher has produced a new drug which has been successful with 11 out of 20 patients. He claims that the new drug is more effective than the standard treatment. Test, at 5% significance level, the claim made by the researcher.

1. Define your test statistic, X and parameter, p.
X is the number of patients in the trial for whom the drug was successful.
p is the probability of success for each patient.

2. Formulate a model for the test statistic.
$X \sim \mathrm{B}(20, p)$

3. Identify your null and alternative hypotheses.
$\mathrm{H}_0: p = 0.4$
$\mathrm{H}_1: p > 0.4$

The researcher claims that the new drug is better so $p > 0.4$

4. Method 1:
Assume H_0 is true and calculate the probability of 11 or more successful treatments
$X \sim \mathrm{B}(20, 0.4)$
$\mathrm{P}(X \geq 11) = 1 - \mathrm{P}(X \leq 10)$
$= 1 - 0.8725$
$= 0.1275$
$= 12.75\%$

5. Compare probability with significance level.
$12.75\% > 5\%$ so, there is not enough evidence to reject H_0

6. Write a conclusion in context.
The new drug is no better than the old one.

OR

Method 2:

1. Work out the critical region and see if 11 lies within it.
$\mathrm{P}(X \geq 13) = 1 - \mathrm{P}(X \leq 12)$
$= 0.021$
$\mathrm{P}(X \geq 12) = 1 - \mathrm{P}(X \leq 11)$
$= 0.0565$
The critical region is 13 or more. Since 11 is not in the critical region, we accept H_0.

2. Write a conclusion in context of the question.
There is no evidence that the new drug is better than the old one.

Two-tailed Test

A two-tailed test is used to test if the probability is changed in either direction. The critical region is split at either end of distribution. The significance level at each end is halved.

For two-tailed tests,

$$\mathrm{H}_1: p \neq \cdots$$

Example 4: In Enrico's restaurant, the ratio of non-vegetarian to vegetarian meals is found to be 2 to 1. In Manuel's restaurant in a random sample of 10 people ordering meals, 1 ordered a vegetarian meal. Using a 5% significance level, test whether the proportion of people eating vegetarian meals in Manuel's restaurant is different from Enrico's restaurant.

1. The proportion of people eating vegetarian meals at Enrico's is $\frac{1}{3}$.

2. X is the number of people in the sample at Manuel's restaurant who ordered vegetarian meals.
p is the probability that a randomly chosen person at Manuel's orders a vegetarian meal.

3. $\mathrm{H}_0: p = \frac{1}{3}$, $\mathrm{H}_1: p \neq \frac{1}{3}$

If H_0 is true, $X \sim \mathrm{B}\left(10, \frac{1}{3}\right)$

4. Method 1:
$\mathrm{P}(X \leq 1) = \mathrm{P}(X = 0) + \mathrm{P}(X = 1)$
$= \left(\frac{2}{3}\right)^{10} + 10\left(\frac{2}{3}\right)^{9}\left(\frac{1}{3}\right)$
$= 0.01734\ldots + 0.08670\ldots$
$= 0.104$ (3s.f.)

0.104 > 0.025 so insufficient evidence to reject H_0.

Method 2:
Let c_1 and c_2 be the two critical values.
$\mathrm{P}(X \leq c_1) \leq 0.025$ and $\mathrm{P}(X \geq c_2) \leq 0.025$

For lower tail:
$\mathrm{P}(X \leq 0) = 0.017341\ldots < 0.025$
$\mathrm{P}(X \leq 1) = 0.10404\ldots > 0.025$
So $c_1 = 0$

For upper tail:
$\mathrm{P}(X \geq 6) = 1 - \mathrm{P}(X \leq 5)$
$= 0.07656\ldots > 0.025$
$\mathrm{P}(X \geq 7) = 1 - \mathrm{P}(X \leq 6)$
$= 0.01966\ldots < 0.025$
So $c_2 = 7$

Observed value of 1 is not in critical region so H_0 is not rejected.

5. Conclusion: There is no evidence that proportion of vegetarian meals at Manuel's restaurant is different to Enrico's.

Questions

Q1.

A lake contains three different types of carp.

There are an estimated 450 mirror carp, 300 leather carp and 850 common carp.

Tim wishes to investigate the health of the fish in the lake.

He decides to take a sample of 160 fish.

(a) Give a reason why stratified random sampling cannot be used.

(1)

(b) Explain how a sample of size 160 could be taken to ensure that the estimated populations of each

type of carp are fairly represented.
You should state the name of the sampling method used.

(2)

As part of the health check, Tim weighed the fish.

His results are given in the table below.

Weight (w kg)	Frequency (f)	Midpoint (m kg)
$2 \leqslant w < 3.5$	8	2.75
$3.5 \leqslant w < 4$	32	3.75
$4 \leqslant w < 4.5$	64	4.25
$4.5 \leqslant w < 5$	40	4.75
$5 \leqslant w < 6$	16	5.5

(You may use $\sum fm = 692$ and $\sum fm^2 = 3053$)

(c) Calculate an estimate for the standard deviation of the weight of the carp.

(2)

Tim realised that he had transposed the figures for 2 of the weights of the fish.

He had recorded in the table 2.3 instead of 3.2 and 4.6 instead of 6.4.

(d) Without calculating a new estimate for the standard deviation, state what effect

(i) using the correct figure of 3.2 instead of 2.3
(ii) using the correct figure of 6.4 instead of 4.6
would have on your estimated standard deviation.
Give a reason for each of your answers.

(2)

(Total for question = 7 marks)

Q2.

Helen is studying one of the qualitative variables from the large data set for Heathrow from 2015.

She started with the data from 3rd May and then took every 10th reading.

There were only 3 different outcomes with the following frequencies

Outcome	A	B	C
Frequency	16	2	1

(a) State the sampling technique Helen used.

(1)

(b) From your knowledge of the large data set

(i) suggest which variable was being studied,
(ii) state the name of outcome A.

(2)

George is also studying the same variable from the large data set for Heathrow from 2015. He started with the data from 5th May and then took every 10th reading and obtained the following

Outcome	A	B	C
Frequency	16	1	1

Helen and George decided they should examine all of the data for this variable for Heathrow from 2015 and obtained the following

Outcome	A	B	C
Frequency	155	26	3

(c) State what inference Helen and George could reliably make from their original samples about the outcomes of this variable at Heathrow, for the period covered by the large data set in 2015.

(1)

(Total for question = 4 marks)

Q3.

Charlie is studying the time it takes members of his company to travel to the office.
He stands by the door to the office from 08 40 to 08 50 one morning and asks workers, as they arrive, how long their journey was.

(a) State the sampling method Charlie used.

(1)

(b) State and briefly describe an alternative method of non-random sampling Charlie could have used to obtain a sample of 40 workers.

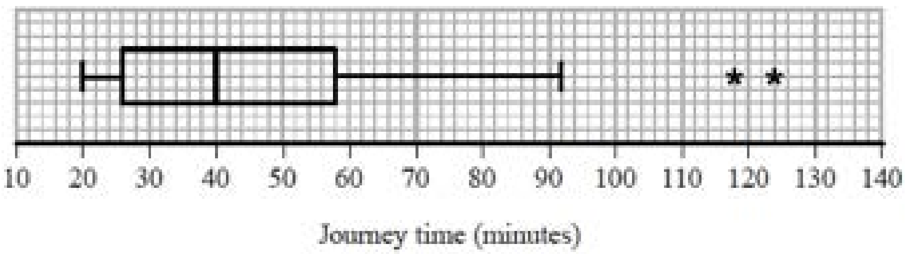
(2)

Taruni decided to ask every member of the company the time, x minutes, it takes them to travel to the office.

(c) State the data selection process Taruni used.

(1)

Taruni's results are summarised by the box plot and summary statistics below.



$$n = 95 \quad \sum x = 4133 \quad \sum x^2 = 202\,294$$

(d) Write down the interquartile range for these data.

(1)

(e) Calculate the mean and the standard deviation for these data.

(3)

(f) State, giving a reason, whether you would recommend using the mean and standard deviation or the median and interquartile range to describe these data.

(2)

Rana and David both work for the company and have both moved house since Taruni collected her data.

Rana's journey to work has changed from 75 minutes to 35 minutes and David's journey to work has changed from 60 minutes to 33 minutes.

Taruni drew her box plot again and only had to change two values.

(g) Explain which two values Taruni must have changed and whether each of these values has increased or decreased.

(3)

(Total for question = 13 marks)

Q4.

(a) State one disadvantage of using quota sampling compared with simple random sampling.

(1)

In a university 8% of students are members of the university dance club.

A random sample of 36 students is taken from the university.

The random variable X represents the number of these students who are members of the dance club.

(b) Using a suitable model for X, find

(i) $P(X = 4)$
(ii) $P(X \geq 7)$

(3)

Only 40% of the university dance club members can dance the tango.

(c) Find the probability that a student is a member of the university dance club and can dance the tango.

(1)

A random sample of 50 students is taken from the university.

(d) Find the probability that fewer than 3 of these students are members of the university dance club and can dance the tango.

(2)

(Total for question = 7 marks)

Mark Scheme

Q1.

Question	Scheme	Marks	AOs
(a)	It is not possible to have a sampling frame	B1	2.3
		(1)	
(b)	Quota sampling **and** (catch 85 common carp, 45 mirror carp and 30 leather carp) **or** (ignore any fish caught of a type where the quota is full)	M1	1.1a
	Quota sampling **and** catch 85 common carp, 45 mirror carp and 30 leather carp **and** ignore any fish caught of a type where the quota is full	A1	1.1b
		(2)	
(c)	$\sigma = \sqrt{\frac{3053}{160} - \left(\frac{692}{160}\right)^2}$	M1	1.1b
	$= 0.6129\ldots$ awrt 0.613	A1	1.1b
		(2)	
(d)(i)	This would have no effect as the piece of data would remain in the same class	B1	2.2a
(ii)	This would increase the standard deviation as change in mean is small and $6.4 - 4.6 \approx 3\sigma$ therefore estimate of standard deviation will increase	B1	2.2a
		(2)	
			(7 marks)

Notes

(a)	B1:	For the idea there cannot be a sampling frame/list
(b)	M1:	Quota sampling **and** either for the correct numbers of each type or for the idea that if quota full ignore the fish.
	A1:	Quota sampling **and** both the correct numbers of each type **and** for the idea that if quota full ignore the fish or sample until all quotas are full
(c)	M1:	A correct expression for σ
	A1:	Awrt 0.613 allow s = awrt 0.615
(d)	B1:	Correct deduction with suitable explanation Allow range for class. Do not allow there is no differences
	B1:	Correct deduction with suitable explanation. so would increase the standard deviation and a suitable reason. Allow the value is bigger than any others in the table oe

Q2.

Qu	Scheme	Marks	AO
(a)	Systematic (sampling)	B1	1.2
		(1)	
(b)(i)	[Daily Mean] Wind Speed	B1	2.2a
(ii)	Light	B1	1.2
		(2)	
(c)	Variable A occurs most (around 80–90%) of the time	B1	2.2b
		(1)	
		(4 marks)	

	Notes
(a)	B1 for identifying the correct sampling technique Allow slight misspelling e.g. "sysmatic", "sytmatic" Do NOT allow "systemic"
(b)(i)	B1 for identifying appropriate qualitative variable. {LDS mark} Allow "Wind speed" or "Wind strength" but NOT just "wind" or "wind direction"
(ii)	B1 for realising that modal wind speed is "Light" {LDS mark} Allow just "light" or "most light"
NB	These two B marks are independent so can score B0B1 for e.g. "rainfall" and "light"
(c)	B1 for inferring that frequency of A can be estimated fairly reliably: {underestimates B and over estimates C} e.g. "A is the most frequent" [can then ignore comments about B and C]

Q3.

Qu	Scheme	Marks	AO
(a)	Convenience or opportunity [sampling]	B1	1.2
		(1)	
(b)	Quota [sampling]	B1	1.1a
	e.g. Take 4 people every 10 minutes	B1	1.1b
		(2)	
(c)	Census	B1	1.2
		(1)	
(d)	[58 − 26 =] <u>32</u> (min)	B1	1.1b
		(1)	
(e)	$\mu = \frac{4133}{95} = 43.505263...$ awrt **<u>43.5</u> (min)**	B1	1.1b
	$\sigma_x = \sqrt{\frac{202\,294}{95} - \mu^2} = \sqrt{236.7026...}$	M1	1.1b
	$= 15.385...$ awrt **<u>15.4</u> (min)**	A1	1.1b
		(3)	
(f)	There are outliers in the data (or data is skew) which will affect mean and sd	B1	2.4
	Therefore use median and IQR	dB1	2.4
		(2)	
(g)	Value of 20, LQ at 26 and outliers will not change <u>or</u> state that median and upper quartile are the values that <u>do</u> change	B1	1.1b
	<u>More values now below 40 than above</u> so Q_2 <u>or</u> Q_3 will change and be lower	M1	2.1
	Both Q_2 <u>and</u> Q_3 will be lower	A1	2.4
		(3)	
		(13 marks)	

	Notes
(b)	1st B1 for quota (sampling) mentioned ("Stratified" or "systematic" or "random" are B0B0) 2nd B1 for a description of how such a system might work, requires suitable strata or categories e.g. time slots, departments, gender, age groups, distance travelled etc Suggestion of randomness is B0
(e)	B1 for a correct mean (awrt 43.5) M1 for a correct expression for the sd (including $\sqrt{\ }$)ft their mean A1 for awrt 15.4 (Allow s = 15.4667… awrt 15.5)
(f)	1st B1 for acknowledging outliers or skewness are a problem for mean and sd "extreme values"/"anomalies" OK. May be implied by saying median and IQR not affected by.. We need to see mention of "outliers", "skewness" and the problem so "data is skewed so use median and IQR" is B0 unless mention that they are not affected by extreme values or mean and standard deviation can be "inflated" by the positive skew etc 2nd dB1 dep on 1st B1 for therefore choosing median and IQR
(g)	B1 for identifying 2 of these 3 groups of unchanged values or stating only Q_2 and Q_3 change M1 for explaining that median or UQ should be lower. E.g. the 2 values have moved to below 40 (or 58) and therefore more than 50% below 40 or (more than 75% below 58) or an argument to show that the other 3 values are the same. (o.e.) Allow arrows on box plot provided statement in words about increased % below 40 or 58 etc A1 for stating median and UQ are both lower with clear evidence of M1 scored [If lots of values on 40 then median might not change but, since two values do change then UQ would change. If this meant that 92 became an outlier then we would have a new value for upper whisker and an extra outlier so effectively 3 values are altered. So median changes]

Q4.

	Scheme	Marks	AO
(a)	**Disadvantage:** e.g. Not random; cannot use (reliably) for inferences	B1	1.1b
		(1)	
(b)	[Sight or correct use of] $X \sim B(36, 0.08)$	M1	3.3
(i)	$P(X = 4) = 0.167387...$ awrt <u>0.167</u>	A1	1.1b
(ii)	$[P(X \geqslant 7) = 1 - P(X \leqslant 6) =]$ $0.022233...$ awrt <u>0.0222</u>	A1	1.1b
		(3)	
(c)	P(In dance club and dance tango) = 0.4×0.08 = <u>0.032</u> or $\frac{4}{125}$ or <u>3.2%</u>	B1	1.1b
		(1)	
(d)	[Let T = those who can dance the Tango. Sight or use of] $T \sim B(50, \text{"}0.032\text{"})$	M1	3.3
	$[P(T < 3) = P(T \leqslant 2) =]$ $0.7850815...$ awrt <u>0.785</u>	A1	1.1b
		(2)	
		(7 marks)	

Notes

(a) B1 for a suitable disadvantage:

Allow (B1)	Do NOT allow (B0)
Not random <u>or</u> less random (o.e.)	Not representative
Cannot use (reliably) for inferences	Less accurate
(More likely to be) biased	Any comment based on time or cost
	Any mention of skew
	Any mention of non-response

(b) M1 for sight of B(36, 0.08) Allow in words: <u>binomial</u> with <u>$n = 36$</u> and <u>$p = 0.08$</u>
may be implied by one correct answer to 2sf <u>or</u> sight of $P(X \leqslant 6) = 0.97776...$ i.e. awrt 0.98
Allow for $36C4 \times 0.08^4 \times 0.92^{32}$ as this is "correct use"

(i) 1st A1 for awrt 0.167 NB An answer of just awrt 0.167 scores M1($\Rightarrow$)1st A1

(ii) 2nd A1 for awrt 0.0222

(c) B1 for 0.032 o.e. (Can allow for sight of 0.4×0.08)

(d) M1 for sight of B(50, "0.032") ft their answer to (c) provided it is a probability $\neq 0.08$
may be implied by correct answer
<u>or</u> sight of $[P(T \leqslant 3)] = 0.924348...$ i.e. awrt 0.924 or $P(T \leqslant 2)$ as part of $1 - P(T \leqslant 2)$ calc.
A1 for awrt 0.785

MR Allow MR of 50 (e.g. 30) provided clearly attempting $P(T \leqslant 2)$ and score M1A0

Year 2: A Level Mathematics	
<u>Statistics: Data Representation and Interpretation</u>	
Self-Assessment: Please identify areas in which you believe are your strong points and those you feel you need to improve on Provide evidence to support your assessment with reference to the content in this booklet.	
Strengths	**Areas for Improvement**

Questions

Q1.

Sara is investigating the variation in daily maximum gust, *t* kn, for Camborne in June and July 1987.

She used the large data set to select a sample of size 20 from the June and July data for 1987. Sara selected the first value using a random number from 1 to 4 and then selected every third value after that.

(a) State the sampling technique Sara used.

(1)

(b) From your knowledge of the large data set, explain why this process may not generate a sample of size 20.

(1)

The data Sara collected are summarised as follows

$$n = 20 \qquad \sum t = 374 \qquad \sum t^2 = 7600$$

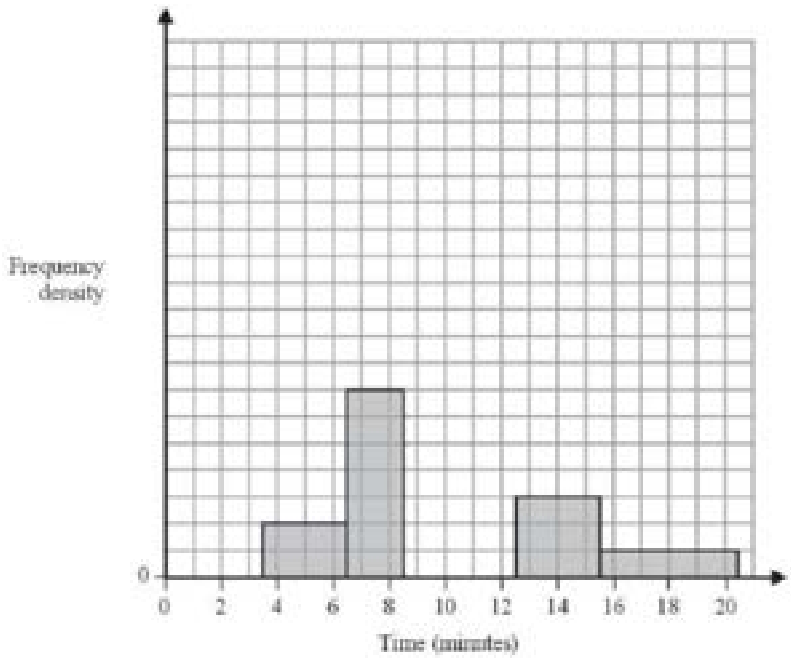
(c) Calculate the standard deviation.

(2)

(Total for question = 4 marks)

Q2.

The partially completed histogram and the partially completed table show the time, to the nearest minute, that a random sample of motorists were delayed by roadworks on a stretch of motorway.



Delay (minutes)	Number of motorists
4 – 6	6
7 – 8	
9	17
10 – 12	45
13 – 15	9
16 – 20	

Estimate the percentage of these motorists who were delayed by the roadworks for between 8.5 and 13.5 minutes.

(5)

(Total for question = 5 marks)

Q3.

Sara was studying the relationship between rainfall, r mm, and humidity, h %, in the UK. She takes a random sample of 11 days from May 1987 for Leuchars from the large data set.

She obtained the following results.

h	93	86	95	97	86	94	97	97	87	97	86
r	1.1	0.3	3.7	20.6	0	0	2.4	1.1	0.1	0.9	0.1

Sara examined the rainfall figures and found

$$Q_1 = 0.1 \qquad Q_2 = 0.9 \qquad Q_3 = 2.4$$

A value that is more than 1.5 times the interquartile range (IQR) above Q_3 is called an outlier.

(a) Show that $r = 20.6$ is an outlier.

(1)

(b) Give a reason why Sara might

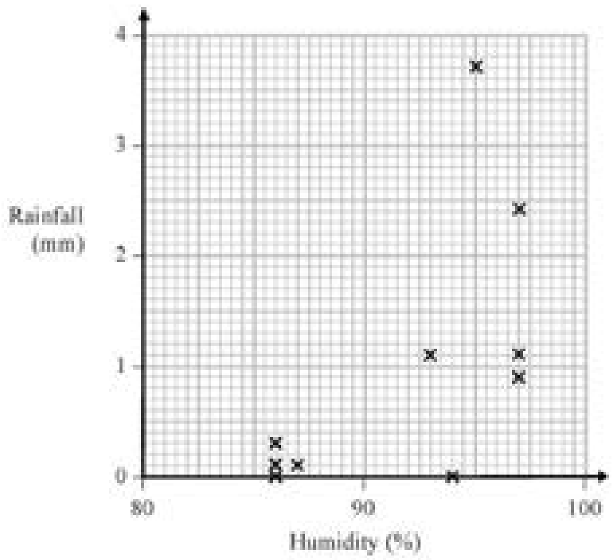
(i) include

(ii) exclude

this day's reading.

(2)

Sara decided to exclude this day's reading and drew the following scatter diagram for the remaining 10 days' values of r and h.



(c) Give an interpretation of the correlation between rainfall and humidity.

(1)

The equation of the regression line of r on h for these 10 days is $r = –12.8 + 0.15h$

(d) Give an interpretation of the gradient of this regression line.

(1)

(e) (i) Comment on the suitability of Sara's sampling method for this study.

(ii) Suggest how Sara could make better use of the large data set for her study.

(2)

(Total for question = 7 marks)

Q4.

Helen is studying the daily mean wind speed for Camborne using the large data set from 1987.
The data for one month are summarised in Table 1 below.

Windspeed	n/a	6	7	8	9	11	12	13	14	16
Frequency	13	2	3	2	2	3	1	2	1	2

Table 1

(a) Calculate the mean for these data.

(1)

(b) Calculate the standard deviation for these data and state the units.

(2)

The means and standard deviations of the daily mean wind speed for the other months from the large data set for Camborne in 1987 are given in Table 2 below. The data are not in month order.

Month	*A*	*B*	*C*	*D*	*E*
Mean	7.58	8.26	8.57	8.57	11.57
Standard Deviation	2.93	3.89	3.46	3.87	4.64

Table 2

(c) Using your knowledge of the large data set, suggest, giving a reason, which month had a mean of 11.57

(2)

The data for these months are summarised in the box plots on the opposite page.
They are not in month order or the same order as in Table 2.

(d) (i) State the meaning of the * symbol on some of the box plots.

(ii) Suggest, giving your reasons, which of the months in Table 2 is most likely to be summarised in the box plot marked *Y*.

(3)

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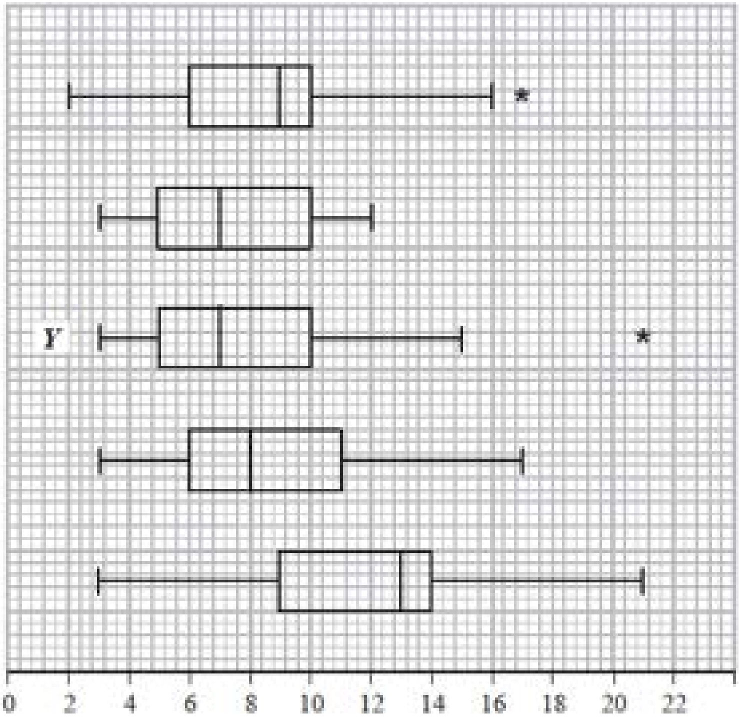


(Total for question = 8 marks)

Q5.

Joshua is investigating the daily total rainfall in Hurn for May to October 2015

Using the information from the large data set, Joshua wishes to calculate the mean of the daily total rainfall in Hurn for May to October 2015

(a) Using your knowledge of the large data set, explain why Joshua needs to clean the data before calculating the mean.

(1)

Using the information from the large data set, he produces the grouped frequency table below.

Daily total rainfall (r mm)	Frequency	Midpoint (x mm)
$0 \leqslant r < 0.5$	121	0.25
$0.5 \leqslant r < 1.0$	10	0.75
$1.0 \leqslant r < 5.0$	24	3.0
$5.0 \leqslant r < 10.0$	12	7.5
$10.0 \leqslant r < 30.0$	17	20.0

You may use $\sum fx = 539.75$ and $\sum fx^2 = 7704.1875$

(b) Use linear interpolation to calculate an estimate for the upper quartile of the daily total rainfall.

(2)

(c) Calculate an estimate for the standard deviation of the daily total rainfall in Hurn for May to October 2015

(2)

(d) (i) State the assumption involved with using class midpoints to calculate an estimate of a mean from a grouped frequency table.

(ii) Using your knowledge of the large data set, explain why this assumption does not hold in this case.

(iii) State, giving a reason, whether you would expect the actual mean daily total rainfall in Hurn for May to October 2015 to be larger than, smaller than or the same as an estimate based on the grouped frequency table.

(3)

(Total for question = 8 marks)

Q6.

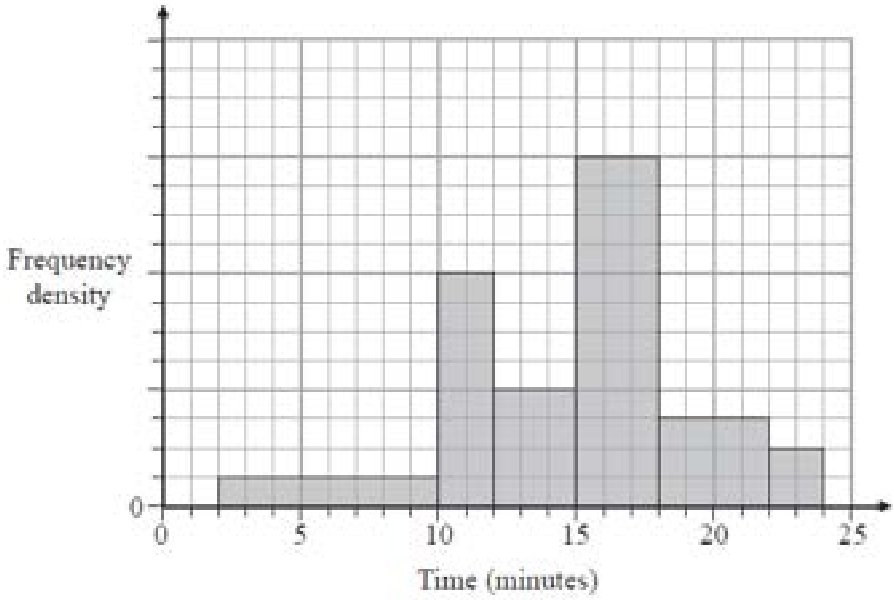


Figure 1

The histogram in Figure 1 shows the times taken to complete a crossword by a random sample of students.

The number of students who completed the crossword in more than 15 minutes is 78.

Estimate the percentage of students who took less than 11 minutes to complete the crossword.

(Total for question = 4 marks)

Q7.

Jerry is studying visibility for Camborne using the large data set June 1987.

The table below contains two extracts from the large data set.

It shows the daily maximum relative humidity and the daily mean visibility.

Date	Daily Maximum Relative Humidity	Daily Mean Visibility
Units	%	
10/06/1987	90	5300
28/06/1987	100	0

(The units for Daily Mean Visibility are deliberately omitted.)

Given that daily mean visibility is given to the nearest 100,

(a) write down the range of distances in metres that corresponds to the recorded value 0 for the daily mean visibility.

(1)

Jerry drew the following scatter diagram, Figure 2, and calculated some statistics using the June 1987 data for Camborne from the large data set.

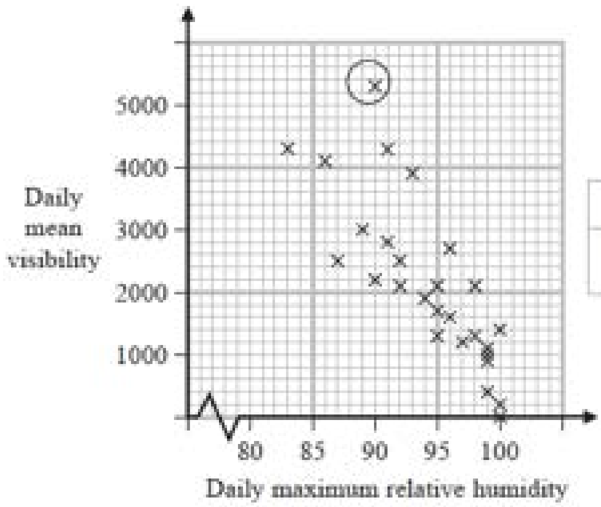


Figure 2

	Q_1	IQR
Daily mean visibility	1100	1600
Daily maximum relative humidity (%)	92	8

Jerry defines an outlier as a value that is more than 1.5 times the interquartile range above Q_3 or more than 1.5 times the interquartile range below Q_1.

(b) Show that the point circled on the scatter diagram is an outlier for visibility.

(2)

(c) Interpret the correlation between the daily mean visibility and the daily maximum relative humidity.

(1)

Jerry drew the following scatter diagram, Figure 3, using the June 1987 data for Camborne from the large data set, but forgot to label the *x*–axis.

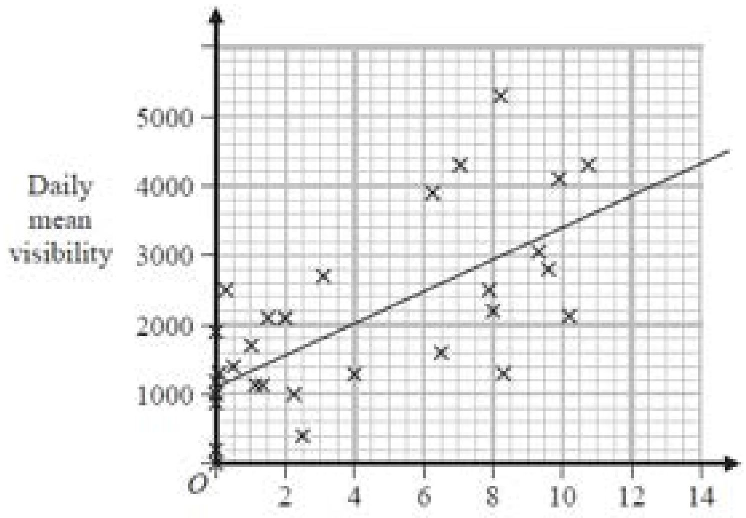


Figure 3

(d) Using your knowledge of the large data set, suggest which variable the *x*-axis on this scatter diagram represents.

(1)

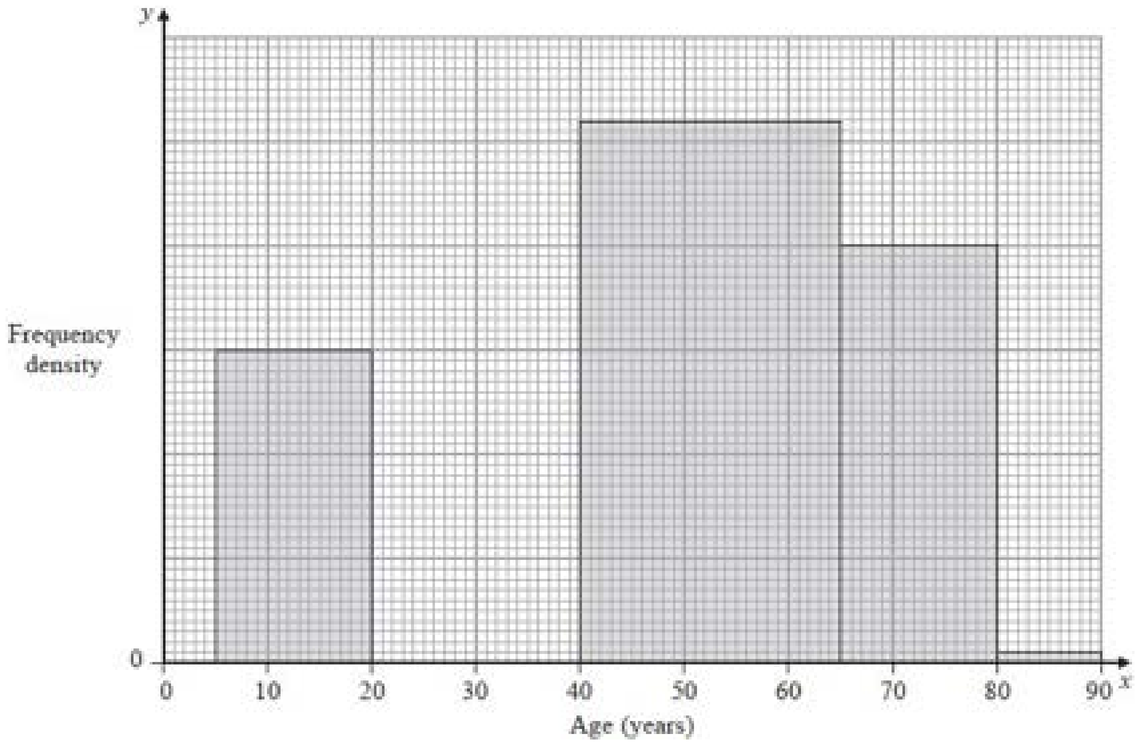
(Total for question = 5 marks)

Q8.

The partially completed table and partially completed histogram give information about the ages of passengers on an airline.

There were no passengers aged 90 or over.

Age (x years)	$0 \leqslant x < 5$	$5 \leqslant x < 20$	$20 \leqslant x < 40$	$40 \leqslant x < 65$	$65 \leqslant x < 80$	$80 \leqslant x < 90$
Frequency	5	45	90			1



(a) Complete the histogram.

(3)

(b) Use linear interpolation to estimate the median age.

(4)

An outlier is defined as a value greater than Q_3 + 1.5 × interquartile range.

Given that Q_1 = 27.3 and Q_3 = 58.9

(c) determine, giving a reason, whether or not the oldest passenger could be considered as an outlier.

(2)

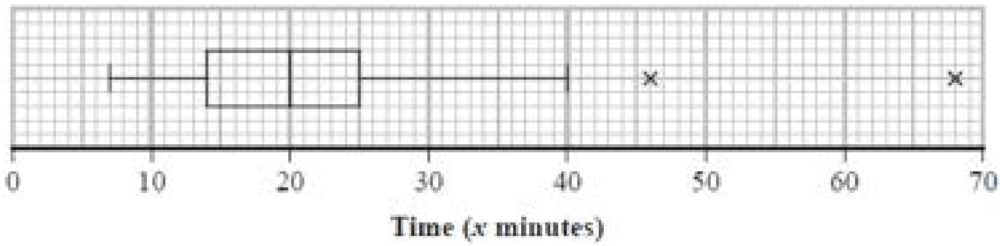
(Total for question = 9 marks)

Q9.

Each member of a group of 27 people was timed when completing a puzzle.

The time taken, *x* minutes, for each member of the group was recorded.

These times are summarised in the following box and whisker plot.



(a) Find the range of the times.

(1)

(b) Find the interquartile range of the times.

(1)

For these 27 people $\sum x = 607.5$ and $\sum x^2 = 17623.25$

(c) calculate the mean time taken to complete the puzzle,

(1)

(d) calculate the standard deviation of the times taken to complete the puzzle.

(2)

Taruni defines an outlier as a value more than 3 standard deviations above the mean.

(e) State how many outliers Taruni would say there are in these data, giving a reason for your answer.

(1)

Adam and Beth also completed the puzzle in *a* minutes and *b* minutes respectively, where *a* > *b*.

When their times are included with the data of the other 27 people

- the median time increases
- the mean time does not change

(f) Suggest a possible value for *a* and a possible value for *b*, explaining how your values satisfy the above conditions.

(3)

(g) Without carrying out any further calculations, explain why the standard deviation of all 29 times will be lower

than your answer to part (d).

(1)

(Total for question = 10 marks)

Q10.

Stav is studying the large data set for September 2015

He codes the variable Daily Mean Pressure, x, using the formula $y = x - 1010$

The data for all 30 days from Hurn are summarised by

$$\sum y = 214 \qquad \sum y^2 = 5912$$

(a) State the units of the variable x

(1)

(b) Find the mean Daily Mean Pressure for these 30 days.

(2)

(c) Find the standard deviation of Daily Mean Pressure for these 30 days.

(3)

Stav knows that, in the UK, winds circulate

- in a **clockwise** direction around a region of **high** pressure
- in an **anticlockwise** direction around a region of **low** pressure

The table gives the Daily Mean Pressure for 3 locations from the large data set on 26/09/2015

Location	Heathrow	Hurn	Leuchars
Daily Mean Pressure	1029	1028	1028
Cardinal Wind Direction			

The Cardinal Wind Directions for these 3 locations on 26/09/2015 were, in random order,

W NE E

You may assume that these 3 locations were under a single region of pressure.

(d) Using your knowledge of the large data set, place each of these Cardinal Wind Directions in the correct location in the table.
Give a reason for your answer.

(2)

(Total for question = 8 marks)

Q11.

Charlie is studying the time it takes members of his company to travel to the office.
He stands by the door to the office from 08 40 to 08 50 one morning and asks workers, as they arrive, how long their journey was.

(a) State the sampling method Charlie used.

(1)

(b) State and briefly describe an alternative method of non-random sampling Charlie could have used to obtain a sample of 40 workers.

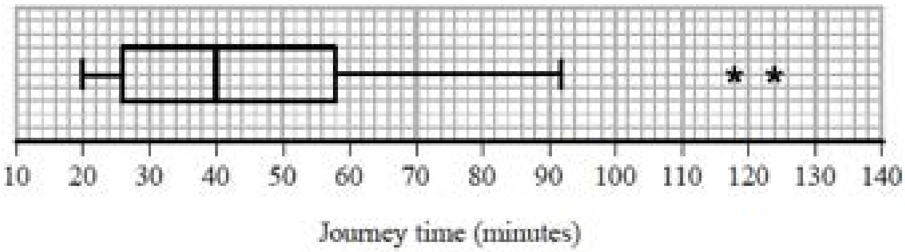
(2)

Taruni decided to ask every member of the company the time, x minutes, it takes them to travel to the office.

(c) State the data selection process Taruni used.

(1)

Taruni's results are summarised by the box plot and summary statistics below.



$$n = 95 \quad \sum x = 4133 \quad \sum x^2 = 202\,294$$

(d) Write down the interquartile range for these data.

(1)

(e) Calculate the mean and the standard deviation for these data.

(3)

(f) State, giving a reason, whether you would recommend using the mean and standard deviation or the median and interquartile range to describe these data.

(2)

Rana and David both work for the company and have both moved house since Taruni collected her data.

Rana's journey to work has changed from 75 minutes to 35 minutes and David's journey to work has changed from 60 minutes to 33 minutes.

Taruni drew her box plot again and only had to change two values.

(g) Explain which two values Taruni must have changed and whether each of these values has increased or decreased.

(3)

(Total for question = 13 marks)

Q12.

A lake contains three different types of carp.

There are an estimated 450 mirror carp, 300 leather carp and 850 common carp.

Tim wishes to investigate the health of the fish in the lake.

He decides to take a sample of 160 fish.

(a) Give a reason why stratified random sampling cannot be used.

(1)

(b) Explain how a sample of size 160 could be taken to ensure that the estimated populations of each

type of carp are fairly represented.
You should state the name of the sampling method used.

(2)

As part of the health check, Tim weighed the fish.

His results are given in the table below.

Weight (w kg)	Frequency (f)	Midpoint (m kg)
$2 \leqslant w < 3.5$	8	2.75
$3.5 \leqslant w < 4$	32	3.75
$4 \leqslant w < 4.5$	64	4.25
$4.5 \leqslant w < 5$	40	4.75
$5 \leqslant w < 6$	16	5.5

(You may use $\sum fm = 692$ and $\sum fm^2 = 3053$)

(c) Calculate an estimate for the standard deviation of the weight of the carp.

(2)

Tim realised that he had transposed the figures for 2 of the weights of the fish.

He had recorded in the table 2.3 instead of 3.2 and 4.6 instead of 6.4.

(d) Without calculating a new estimate for the standard deviation, state what effect

(i) using the correct figure of 3.2 instead of 2.3
(ii) using the correct figure of 6.4 instead of 4.6
would have on your estimated standard deviation.
Give a reason for each of your answers.

(2)

(Total for question = 7 marks)

Mark Scheme

Q1.

Question	Scheme	Marks	AOs
(a)	Systematic (sample)	B1cao	1.2
(b)	In LDS some days have gaps because the data was not recorded	B1	2.4
(c)	$\left[\bar{t} = \frac{374}{20} = 18.7\right]$ $\sigma_t = \sqrt{\frac{7600}{20} - \bar{t}^2}$ $[=\sqrt{30.31}]$	M1	1.1a
	$= 5.5054\ldots$ awrt $\underline{5.51}$ (Accept use of $s_t = \sqrt{\frac{7600 - 20\bar{t}^2}{19}} = 5.6484\ldots$)	A1	1.1b
			(4 marks)

Part	Notes
(b)	B1 a correct explanation
(c)	M1 for a correct expression for $\bar{t}$ and σ_t or s_t. Ft an incorrect evaluation of $\bar{t}$
	A1 for σ_t = awrt 5.51 or s_t = awrt 5.65

Q2.

Question	Scheme	Marks	AOs
	$17 + 45 + \frac{1}{3} \times 9$ $[= 65]$	M1	2.2a
	$(7 - 8)$ $\underline{14}$ or $(16 - 20)$ $\underline{5}$ [Values may be seen in the table]	M1 A1	3.1a 1.1b
	Percentage of motorists is $\frac{\text{"65"}}{6 + \text{"14"} + 17 + 45 + 9 + \text{"5"}} \times 100$	M1	3.1b
	$= \underline{67.7\%}$	A1	1.1b
			(5 marks)

Part	Notes
	1st M1 for a fully correct expression for the number of motorists in the interval
	2nd M1 for clear use of frequency density in (4-6) or (13-15) cases to establish the fd scale. Then use of area to find frequency in one of the missing cases.
	1st A1 for both correct values seen
	3rd M1 for realising that total is required and attempting a correct expression for %
	2nd A1 for awrt 67.7%

Q3.

Question	Scheme	Marks	AOs
(a)	IQR = 2.3 and $20.6 \gg 2.4 + 1.5 \times 2.3$ (= 5.85) (Compare correct values)	B1	1.1b
		(1)	
(b)(i)	e.g. it is a piece of data and we should consider all the data (o.e.)	B1	2.4
(ii)	e.g. it is an extreme value and could unduly influence the analysis or it could be a mistake	B1	2.4
		(2)	
(c)	e.g. "as humidity increases rainfall increases"	B1	2.2b
		(1)	
(d)	e.g. a 10% increase in humidity gives rise to a 1.5 mm increase in rainfall or represents 0.15mm of rainfall per percentage of humidity	B1	3.4
		(1)	
(e)(i)	Not a good method since only uses 11 days from one location in one month.	B1	2.4
(ii)	e.g. She should use data from more of the UK locations and more of the months or using a spreadsheet or computer package she could use all of the available UK data	B1	2.4
		(2)	
		(7 marks)	

Part	Notes
(a)	B1 for sight of the correct calculation and suitable comparison with 20.6
(b)(i)	B1 for a suitable reason for including the data point
(ii)	B1 for a suitable reason for excluding the data point
(c)	B1 for a suitable interpretation of positive correlation mentioning humidity and rainfall
(d)	B1 for a suitable description of the rate: rainfall per percentage of humidity including reference to values.
(e)(i)	B1 for a comment that supports the idea that her sampling method was not a good one
(ii)	B1 for some sensible suggestions that would give a better representation of the data across the UK. Must show some awareness of the fact that LDS has different locations and more months of data available but must be clear they are NOT using any overseas locations. NB B0 for a comment that says use more than one location without specifying that only UK locations are required.

Q4.

Qu	Scheme	Marks	AO
(a)	$\bar{x} = 10.2(2222...)$ awrt **10.2**	B1	1.1b
		(1)	
(b)	$\sigma_x = 3.17(20227...)$ awrt **3.17**	B1ft	1.1b
	Sight of "knots" or "kn" (condone knots/s etc)	B1	1.2
		(2)	
(c)	October ..... since	B1	2.2b
	it is windier in the autumn or month of the hurricane or latest month in the year	B1	2.4
		(2)	
(d)(i)	They represent outliers	B1	1.2
(ii)	Y has low median so expect lowish mean (but outlier so > 7) and	M1	2.4
	Y has big range/IQR or spread so expect larger st.dev		
	Suggests B	A1	2.2b
		(3)	
		(8 marks)	

	Notes
NB	$\bar{x} = \frac{184}{18}$ and $\sigma_x = \sqrt{\frac{2062}{18} - \bar{x}^2}$
(a)	B1 for $\bar{x} = 10.2$ (allow exact fraction)
(b)	1st B1ft allow 3.2 from a correct expr' accept $s = 3.26(3984...)$ [ft use of n/a] Treating n/a as 0 May see $n = 31$ or $\bar{x} = 5.9354...$ which is B0 in (a) but here in (b) it gives $\sigma_x = 5.59(34...)$ or $s = 5.6858...$(awrt 5.69) and scores 1st B1 2nd B1 accept kn accept in (a) or (b) (allow nautical miles/hour)
(c)	1st B1 choosing October but accept September. 2nd B1 for stating that (Camborne) is windier in autumn/winter months "because it is winter/autumn/windier/colder in "month" " Sep $\leqslant$ "month" $\leqslant$ Mar scores B1B1 for "month" = Sep or Oct and B0B1 for other months in range
(d)(i)	B1 for outlier or the idea of an extreme value allow "anomaly"
(ii)	M1 for a comment relating to location that mentions both median and mean and a comment relating to spread that mentions both range/IQR and standard deviation and leads to choosing B, C or D **Choosing A or E is M0** Incorrect/false statements score M0 e.g. Q_3 = (mean + σ) or identify Q_2 = mean or Y has small spread
ALT	**Use of outliers:** outlier is (mean + 3σ) (B = 19.9), (C = 18.95), (D = 20.2) Must see at least one of these values and compare to Y's outlier[leads to D or B] A1 for suitable inference i.e. B (accept D or B or D) M1 must be scored

Q5.

Question	Scheme	Marks	AOs
(a)	Tr(ace) (data needs to be converted to numbers before the calculation can be carried out)	B1	2.4
		(1)	
(b)	$[1+]\frac{138-131}{24}\times 4$	M1	2.1
	$= 2.1666....$ awrt **2.17**	A1	1.1b
		(2)	
(c)	$\sigma = \sqrt{\frac{7704.1875}{184} - \left(\frac{539.75}{184}\right)^2} = 5.7676...$ $\sigma =$ awrt **5.77**	M1 A1	1.1b 1.1b
		(2)	
(d)(i)	Using class midpoints to estimate the mean assumes that the values are uniformly distributed **within the class(es)**.	B1	2.4
(ii)& (iii)	This is not the case here as the majority of the data (in the first class) are 0.	B1	2.3
	The actual mean is likely to be smaller than the estimate (since the first group has more values at 0 and close to 0)	dB1	2.2b
		(3)	
			(8 marks)

Notes	
(a)	**B1:** Identifying tr(ace) data Ignore comments about n/a, missing data, anomalies, etc.
(b)	**M1:** Correct fraction $\frac{7}{24}\times 4$ allow working down $[5]-\frac{155-138}{24}\times 4$ allow a correct equation leading to a correct fraction e.g. $\frac{x-1}{5-1}=\frac{138-131}{155-131}$ for M1 Use of $(n+1)$ with 138.75 allow $\frac{7.75}{24}\times 4$ **A1:** awrt 2.17 (condone $\frac{13}{6}$) awrt 2.29 from $(n+1)$ (condone $\frac{55}{24}$)
(c)	**M1:** Correct expression for standard deviation (allow mean = awrt 2.93) **A1:** awrt 5.77 correct answer only scores M1A1 (allow $s = 5.78$) **SC:** 5.76 with no working scores M1A0
(d)(i)	**B1:** Explaining that data assumed to be spread evenly across each class (o.e.) e.g. The midpoint of each class is the mean of each class or all the values in the class are located at the midpoint condone normally distributed within each class
Mark together (ii)&(iii)	**B1:** Demonstrating an understanding of the LDS that the majority of data values (in the first class) are at 0 or close to 0 (trace). **dB1:** (dependent upon 2nd B1) Correct inference based on knowledge of the LDS **SC:** If B1 is scored in (i) for 'The data are spread evenly across each class,' then in (ii) 'The data are not evenly distributed in the classes' scores B1 but in (iii) 'the actual mean is smaller' with no further justification scores B0

Q6.

Question	Scheme	Marks	AOs
	1 square is $\frac{78}{12\times3+3\times4+2\times2}=\left[\frac{78}{52}=1.5\right]$ and $(8\times1+1\times8)\times$"1.5"	M1	3.1a
	24 students took less than 11 minutes	A1	1.1b
	Percentage of students = $\frac{\text{"24"}}{78+\text{"24"}+1\times8\times\text{"1.5"}+3\times4\times\text{"1.5"}}\times100$	M1	3.1b
	= 18.18… awrt 18%	A1	1.1b
		(4)	
			Total 4

Notes		
	M1:	For clear use of frequency density to establish the fd scale and then use the area to find frequency of <11 minutes. Allow maximum of 3 errors in either the heights or widths in total if working shown. They may calculate the area using other size squares. Allow for realising they need to find the total number of squares (88) maximum of 4 errors in either the heights or widths and number < 11 minutes(16) - must have a maximum of 1 error in either the heights or widths (and not use the 78 as part of calulation)
	A1:	For correct values seen. Allow for 88 and 16
	M1:	For realising the need to find the total and calculating a percentage. (with "their 24" as the numerator). Allow $(8\times1+2\times8)\times$"1.5" instead of "24"$+1\times8\times$"1.5" If working shown can allow maximum of 2 errors in either the heights or widths in the calculation of the total. Allow "their 24" / 132 oe
	A1:	awrt 18

Q7.

Question	Scheme	Marks	AOs
(a)	0 to 500 m	B1	1.2
		(1)	
(b)	$1100+1600+1.5\times1600$ [= 5100]	M1	2.1
	5300 > 5100 therefore outlier	A1	1.1b
		(2)	
(c)	As the humidity increases the mean visibility decreases	B1	2.4
		(1)	
(d)	(Hours of) sunshine	B1	2.2b
		(1)	
			(5 marks)

Notes		
(a)	B1:	For realising it is the maximum distance and distance given with correct units. Allow 0 to 50dm or < 500m or < 50dm
(b)	M1:	Attempt to find Q_3 and the upper limit
	A1:	5100, if a value for the point is stated it must be above 5100 otherwise it is A0. For a statement comparing and conclusion it is an outlier or it is above Q_3 +1.5IQR. Allow accept the point circled is greater than 5100 oe
(c)	B1:	For a suitable interpretation of a negative correlation mentioning humidity and visibility
(d)	B1:	A correct deduction that the unlabelled variable is the hours of sunshine. Condone missing hours. Do not allow if more than one variable given. Must be quantative variable Not cloud cover since values bigger than 8 Not wind speed since values not integers Not daily mean temperature since mean temperature near to zero are unlikely in June

Q8.

Qu	Scheme	Marks	AO
(a)	From [5,20) fd = 3 <u>or</u> 1 large square = 2.5 passengers o.e.	M1	2.2a
	Correct bar above [0, 5)	A1	1.1b
	Correct bar above [20, 40)	A1	1.1b
		(3)	
(b)	For [40, 65) **<u>130</u>** passengers <u>or</u> for [65, 80) **<u>60</u>** passengers	M1	2.1
	For attempt to find total number of passengers = **<u>331</u>**	A1ft	1.1b
	[Median =] $40+\frac{\frac{1}{2}(\text{"331"})-140}{\text{"130"}}\times 25$ <u>or</u> $65-\frac{270-\frac{1}{2}(\text{"331"})}{\text{"130"}}\times 25$ (o.e.)	M1	1.1b
	= 44.9038… = awrt **<u>44.9</u>**	A1	1.1b
		(4)	
(c)	Upper outlier limit = $58.9 + 1.5\times(58.9 - 27.3) = 106\ (.3) > 90$	M1	2.4
	So oldest passenger is <u>not</u> an outlier	A1	2.2a
		(2)	
		(9 marks)	

	Notes
(a)	M1 for attempt at fd or a suitable method to deduce the scale for the histogram May be implied by one correct bar. 1st A1 for first bar [0, 5) with fd = 1 <u>or</u> 2 large squares high 2nd A1 for third bar with fd = 4.5 <u>or</u> 9 large squares high
(b)	1st M1 for an attempt using their fd to find the missing frequencies. May be in table 1st A1ft for a clear attempt to find the total number of passengers (ft their 130 and 60) 2nd M1 for any expression/equation leading to correct Q_2 Must be using 40-65 class 2nd A1 for awrt 44.9 (allow $(n + 1)$ leading to 45)
(c)	M1 for finding the upper outlier limit (expression or awrt 106) <u>and</u> stating or implying > 90 A1 dep on M1 seen for deducing NOT an outlier

Q9.

	Scheme	Marks	AO
(a)	$[68 - 7 =]$ $\underline{61}$ (only)	B1	1.1b
		(1)	
(b)	$[25 - 14] = \underline{\mathbf{11}}$	B1	1.1b
		(1)	
(c)	$\left[\mu \text{ or } \bar{x} = \frac{607.5}{27} = \right] = \underline{\mathbf{22.5}}$	B1	1.1b
		(1)	
(d)	$\sigma = \sqrt{\frac{17\,623.25}{27} - \text{"}22.5\text{"}^2}$ <u>or</u> $\sqrt{146.4629...}$	M1	1.1b
	$= 12.10218...$ awrt $\underline{\mathbf{12.1}}$	A1	1.1b
		(2)	
(e)	$\mu + 3\sigma = \text{"}22.5\text{"} + 3 \times \text{"}12.1...\text{"} =$ awrt 59 so only <u>**one**</u> outlier	B1ft	1.1b
		(1)	
(f)	Median increases implies that both values must be > 20	M1	3.1b
	Mean is the same means that $a + b = 45$	M1	1.1b
	So possible values are: e.g. $b = 21$ and $a = 24$ (o.e.)	A1	2.2b
		(3)	
(g)	Both values will be less than 1 standard deviation from the mean and so the standard deviation of all 29 values will be smaller	B1	2.4
		(1)	
		(10 marks)	

	Notes
(a)	B1 for correctly interpreting the box plot to find the range (more than 1 answer is B0)
(b)	B1 for correct understanding of IQR and answer of 11
(c)	B1 for 22.5 only (or exact equivalent such as $\frac{45}{2}$). Allow 22 mins and 30 secs.
(d)	M1 for a correct expression including square root. Allow $\sqrt{146}$ or better. Ft their mean A1 for awrt 12.1 NB Allow use of $s = 12.3327...$ or awrt 12.3
(e)	B1ft for a correct calculation or value based on their μ and σ and compatible conclusion
(f)	1st M1 Correct start to the problem and a correct statement about the values based on median Allow if their final two values are both >20 2nd M1 for a correct explanation leading to equation $a + b = 45$ (o.e. e.g. equidistant from mean) Allow if their final two values sum to 45 A1 for a correct pair of values (both > 20 with a sum of 45) **and** at least some attempt to explain how their values satisfy at least one of the conditions (both > 20 <u>or</u> $a + b = 45$). Ignore $a =$ or $b =$ labels
NB	The values for a and b do not need to be integers.
(g)	B1 for a correct explanation. Must mention that both values are less than 1 sd (ft their answer to (d)) from the mean

Q10.

	Scheme	Marks	AO
(a)	Hectopascal or hPa	B1 (1)	1.2
(b)	$\bar{x} = \bar{y} + 1010$ or $\frac{214}{30} + 1010$	M1	1.1b
	$= 1017.1333\ldots$ awrt **1017**	A1 (2)	1.1b
(c)	$\sigma_x = \sigma_y$ (or statement that standard deviation is not affected by this type of coding)	M1	3.1b
	$[\sigma_y =] \sqrt{\frac{5912}{30} - (\text{"}7.13[33\ldots]\text{"})^2}$ or $\sqrt{146.1822\ldots}$	M1	1.1b
	$= 12.0905\ldots$ awrt **12.1**	A1 (3)	1.1b
(d)	High pressure (since approx. mean + sd) so clockwise Locations are (from North to South): Leuchars, Heathrow, Hurn	B1	2.4
	Wind direction is direction wind blows from So: Heathrow (NE) Hurn (E) Leuchars (W)	B1 (2)	2.2a
		(8 marks)	

	Notes
FYI	1 hPa = 100 Pa; 10hPa = 1 kPa; 1Pa = 1 Nm^{-2}
(a)	B1 for "hectopascal" or hPa (condone pascals, allow millibars or mb) o.e. Do NOT allow kPa or kilopascals or Pa on its own
(b)	M1 for a strategy to find $\bar{x}$ Allow an attempt to find $\sum x$ that gets as far as $\sum x = \sum y + 30 \times 1010 [= 30\,514]$ A1 for awrt 1017 (accept 1020) [Ignore incorrect units]
(c)	1st M1 for an overall strategy using the fact $\sigma_x = \sigma_y$ (can be implied by correct final ans) or for $\sum x = 30\,514$ and $\sum x^2 = 31\,041\,192$ (both seen and correct) 2nd M1 for a correct expression (with $\sqrt{\ }$)(ft their $\bar{y}$ to 3sf) allow awrt 146 for 146.1822.. or for correct expression in x can ft their $\sum x > 30\,000$ or their answer to (b) A1 (dep on 2nd M1) for awrt 12.1 [Ignore incorrect units]
Final answer	Final ans of awrt 12.1 scores 3/3 but if they then adjust for x e.g. add 1010 (M0M1A1)
(d)	1st B1 for at least one of these reasons (these 2 lines) clearly stated (may see diagram) Need "high pressure" **and** "clockwise" to score on 1st line Contradictory statements B0 e.g. correct N–S list but say "anticlockwise" 2nd B1 (indep of 1st B1) for deducing the 3 correct directions either in the table or stated as above If the answers in table and text are different we take the table (as question says)

Q11.

Qu	Scheme	Marks	AO
(a)	Convenience <u>or</u> opportunity [sampling]	B1	1.2
		(1)	
(b)	Quota [sampling]	B1	1.1a
	e.g. Take 4 people every 10 minutes	B1	1.1b
		(2)	
(c)	Census	B1	1.2
		(1)	
(d)	[58 − 26 =] <u>32</u> (min)	B1	1.1b
		(1)	
(e)	$\mu = \frac{4133}{95} = 43.505263...$ **awrt <u>43.5</u> (min)**	B1	1.1b
	$\sigma_x = \sqrt{\frac{202\,294}{95} - \mu^2} = \sqrt{236.7026...}$	M1	1.1b
	= 15.385... **awrt <u>15.4</u> (min)**	A1	1.1b
		(3)	
(f)	There are outliers in the data (or data is skew) which will affect mean and sd	B1	2.4
	Therefore use median and IQR	dB1	2.4
		(2)	
(g)	Value of 20, LQ at 26 and outliers will not change <u>or</u> state that median and upper quartile are the values that <u>do</u> change	B1	1.1b
	<u>More values now below 40 than above</u> so Q_2 <u>or</u> Q_3 will change and be lower	M1	2.1
	Both Q_2 <u>and</u> Q_3 will be lower	A1	2.4
		(3)	
		(13 marks)	

	Notes
(b)	1st B1 for quota (sampling) mentioned ("Stratified" or "systematic" or "random" are B0B0) 2nd B1 for a description of how such a system might work, requires suitable strata or categories e.g. time slots, departments, gender, age groups, distance travelled etc Suggestion of randomness is B0
(e)	B1 for a correct mean (awrt 43.5) M1 for a correct expression for the sd (including $\sqrt{\ }$)ft their mean A1 for awrt 15.4 (Allow s = 15.4667... awrt 15.5)
(f)	1st B1 for acknowledging <u>outliers</u> or <u>skewness</u> are a problem for <u>mean and sd</u> "extreme values"/"anomalies" OK May be implied by saying median and IQR not affected by.. We need to see mention of "outliers", "skewness" and the problem so "data is skewed so use median and IQR" is B0 unless mention that they are not affected by extreme values <u>or</u> mean and standard deviation can be "inflated" by the positive skew etc 2nd dB1 dep on 1st B1 for therefore choosing <u>median and IQR</u>
(g)	B1 for identifying 2 of these 3 groups of unchanged values or stating only Q_2 and Q_3 change M1 for <u>explaining</u> that median or UQ should be lower. E.g. the 2 values have moved to below 40 (or 58) and therefore more than 50% below 40 or (more than 75% below 58) <u>or</u> an argument to show that the other 3 values are the same. (o.e.) Allow arrows on box plot provided statement in words about increased % below 40 or 58 etc A1 for stating median <u>and</u> UQ are both lower with clear evidence of M1 scored [If lots of values on 40 then median might not change but, since two values <u>do</u> change then UQ would change. If this meant that 92 became an outlier then we would have a new value for upper whisker and an extra outlier so effectively 3 values are altered. So median changes]

Q12.

Question	Scheme	Marks	AOs
(a)	It is not possible to have a sampling frame	B1	2.3
		(1)	
(b)	Quota sampling **and** (catch 85 common carp, 45 mirror carp and 30 leather carp) **or** (ignore any fish caught of a type where the quota is full)	M1	1.1a
	Quota sampling **and** catch 85 common carp, 45 mirror carp and 30 leather carp **and** ignore any fish caught of a type where the quota is full	A1	1.1b
		(2)	
(c)	$\sigma = \sqrt{\frac{3053}{160} - \left(\frac{692}{160}\right)^2}$	M1	1.1b
	$= 0.6129...$ awrt 0.613	A1	1.1b
		(2)	
(d)(i)	This would have no effect as the piece of data would remain in the same class	B1	2.2a
(ii)	This would increase the standard deviation as change in mean is small and $6.4 - 4.6 \approx 3\sigma$ therefore estimate of standard deviation will increase	B1	2.2a
		(2)	
			(7 marks)

Notes		
(a)	B1:	For the idea there cannot be a sampling frame/list
(b)	M1:	Quota sampling **and** either for the correct numbers of each type **or** for the idea that if quota full ignore the fish.
	A1:	Quota sampling **and** both the correct numbers of each type **and** for the idea that if quota full ignore the fish or sample until all quotas are full
(c)	M1:	A correct expression for σ
	A1:	Awrt 0.613 allow s = awrt 0.615
(d)	B1:	Correct deduction with suitable explanation Allow range for class. Do not allow there is no differences
	B1:	Correct deduction with suitable explanation. so would increase the standard deviation and a suitable reason. Allow the value is bigger than any others in the table oe

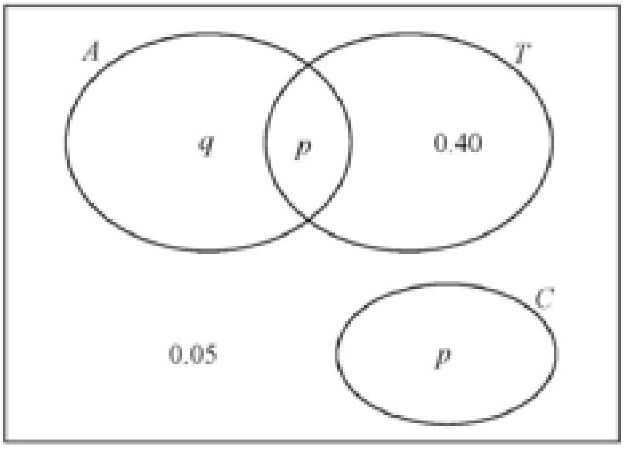
Year 2: A Level Mathematics	
<u>Statistics: Probability</u>	
Self-Assessment: Please identify areas in which you believe are your strong points and those you feel you need to improve on Provide evidence to support your assessment with reference to the content in this booklet.	
Strengths	**Areas for Improvement**

Questions

Q1.

The Venn diagram shows the probabilities for students at a college taking part in various sports.

A represents the event that a student takes part in Athletics.
T represents the event that a student takes part in Tennis.
C represents the event that a student takes part in Cricket.
p and *q* are probabilities.



The probability that a student selected at random takes part in Athletics or Tennis is 0.75

(a) Find the value of *p*.

(1)

(b) State, giving a reason, whether or not the events *A* and *T* are statistically independent. Show your working clearly.

(3)

(c) Find the probability that a student selected at random does not take part in Athletics or Cricket.

(1)

(Total for question = 5 marks)

Q2.

A factory buys 10% of its components from supplier *A*, 30% from supplier *B* and the rest from supplier *C*. It is known that 6% of the components it buys are faulty.

Of the components bought from supplier *A*, 9% are faulty and of the components bought from supplier *B*, 3% are faulty.

(a) Find the percentage of components bought from supplier *C* that are faulty.

(3)

A component is selected at random.

(b) Explain why the event "the component was bought from supplier *B* " is not statistically independent from the event "the component is faulty".

(1)

(Total for question = 4 marks)

Q3.

A biased spinner can only land on one of the numbers 1, 2, 3 or 4. The random variable X represents the number that the spinner lands on after a single spin and $P(X = r) = P(X = r + 2)$ for $r = 1, 2$

Given that $P(X = 2) = 0.35$

(a) find the complete probability distribution of X.

(2)

Ambroh spins the spinner 60 times.

(b) Find the probability that more than half of the spins land on the number 4

Give your answer to 3 significant figures.

(3)

The random variable $Y = \frac{12}{X}$

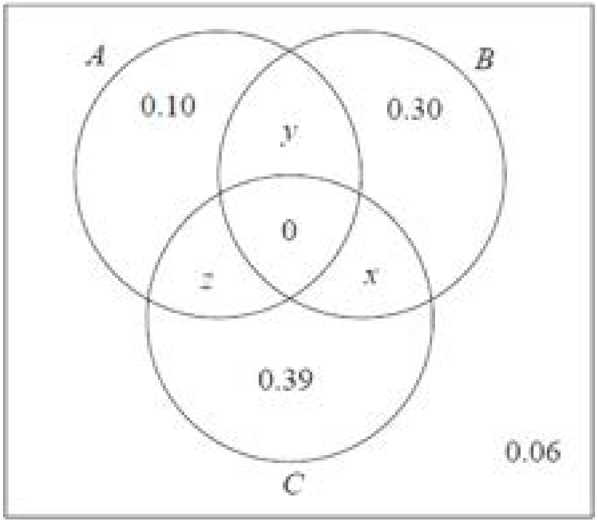
(c) Find $P(Y - X \leq 4)$

(3)

(Total for question = 8 marks)

Q4.

The Venn diagram shows three events, *A*, *B* and *C*, and their associated probabilities.



Events *B* and *C* are mutually exclusive.
Events *A* and *C* are independent.

Showing your working, find the value of x, the value of y and the value of z.

(Total for question = 5 marks)

Q5.

A fair 5-sided spinner has sides numbered 1, 2, 3, 4 and 5

The spinner is spun once and the score of the side it lands on is recorded.

(a) Write down the name of the distribution that can be used to model the score of the side it lands on.

(1)

The spinner is spun 28 times.

The random variable X represents the number of times the spinner lands on 2

(b) (i) Find the probability that the spinner lands on 2 at least 7 times.

(ii) Find $P(4 \leq X < 8)$

(5)

(Total for question = 6 marks)

Q6.

In a game, a player can score 0, 1, 2, 3 or 4 points each time the game is played.

The random variable S, representing the player's score, has the following probability distribution where a, b and c
are constants.

s	0	1	2	3	4
$P(S = s)$	a	b	c	0.1	0.15

The probability of scoring less than 2 points is twice the probability of scoring at least 2 points.

Each game played is independent of previous games played.

John plays the game twice and adds the two scores together to get a total.

Calculate the probability that the total is 6 points.

(Total for question = 6 marks)

Q7.

Afrika works in a call centre.

She assumes that calls are independent and knows, from past experience, that on each sales call

that she makes there is a probability of $\frac{1}{6}$ that it is successful.

Afrika makes 9 sales calls.

(a) Calculate the probability that at least 3 of these sales calls will be successful.

(2)

The probability of Afrika making a successful sales call is the same each day.

Afrika makes 9 sales calls on each of 5 different days.

(b) Calculate the probability that at least 3 of the sales calls will be successful on exactly 1 of these days.

(2)

Rowan works in the same call centre as Afrika and believes he is a more successful salesperson.

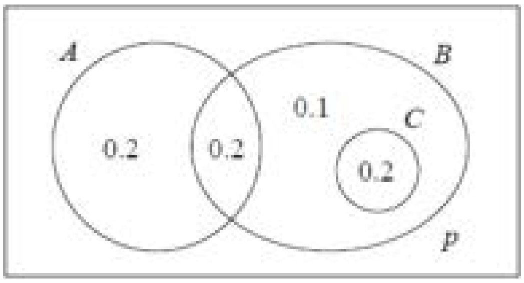
To check Rowan's belief, Afrika monitors the next 35 sales calls Rowan makes and finds that 11 of the sales calls are successful.

(c) Stating your hypotheses clearly test, at the 5% level of significance, whether or not there is evidence to support Rowan's belief.

(4)

(Total for question = 8 marks)

Q8.



The Venn diagram, where p is a probability, shows the 3 events A, B and C with their associated probabilities.

(a) Find the value of p.

(1)

(b) Write down a pair of mutually exclusive events from A, B and C.

(1)

(Total for question = 2 marks)

Q9.

Two bags, **A** and **B**, each contain balls which are either red or yellow or green.

Bag **A** contains 4 red, 3 yellow and n green balls.
Bag **B** contains 5 red, 3 yellow and 1 green ball.

A ball is selected at random from bag **A** and placed into bag **B**.
A ball is then selected at random from bag **B** and placed into bag **A**.

The probability that bag **A** now contains an equal number of red, yellow and green balls is p.

Given that $p > 0$, find the possible values of n and p.

(Total for question = 5 marks)

Q10.

Helen believes that the random variable C, representing cloud cover from the large data set, can be modelled by a discrete uniform distribution.

(a) Write down the probability distribution for C.

(2)

(b) Using this model, find the probability that cloud cover is less than 50%

(1)

Helen used all the data from the large data set for Hurn in 2015 and found that the proportion of days with cloud cover of less than 50% was 0.315

(c) Comment on the suitability of Helen's model in the light of this information.

(1)

(d) Suggest an appropriate refinement to Helen's model.

(1)

(Total for question = 5 marks)

Q11.

Magali is studying the mean total cloud cover, in oktas, for Leuchars in 1987 using data from the large data set. The daily mean total cloud cover for all 184 days from the large data set is summarised in the table below.

Daily mean total cloud cover (oktas)	0	1	2	3	4	5	6	7	8
Frequency (number of days)	0	1	4	7	10	30	52	52	28

One of the 184 days is selected at random.

(a) Find the probability that it has a daily mean total cloud cover of 6 or greater.

(1)

Magali is investigating whether the daily mean total cloud cover can be modelled using a binomial distribution.

She uses the random variable X to denote the daily mean total cloud cover and believes that $X \sim B(8, 0.76)$

Using Magali's model,

(b) (i) find $P(X \geq 6)$

(2)

(ii) find, to 1 decimal place, the expected number of days in a sample of 184 days with a daily mean total cloud cover of 7

(2)

(c) Explain whether or not your answers to part (b) support the use of Magali's model.

(1)

There were 28 days that had a daily mean total cloud cover of 8
For these 28 days the daily mean total cloud cover for the **following** day is shown in the table below.

Daily mean total cloud cover (oktas)	0	1	2	3	4	5	6	7	8
Frequency (number of days)	0	0	1	1	2	1	5	9	9

(d) Find the proportion of these days when the daily mean total cloud cover was 6 or greater.

(1)

(e) Comment on Magali's model in light of your answer to part (d).

(2)

(Total for question = 9 marks)

Q12.

The discrete random variable D has the following probability distribution

d	10	20	30	40	50
$P(D = d)$	$\frac{k}{10}$	$\frac{k}{20}$	$\frac{k}{30}$	$\frac{k}{40}$	$\frac{k}{50}$

where k is a constant.

(a) Show that the value of k is $\frac{600}{137}$

(2)

The random variables D_1 and D_2 are independent and each have the same distribution as D.

(b) Find P ($D_1 + D_2 = 80$)

Give your answer to 3 significant figures.

(3)

A single observation of D is made.

The value obtained, d, is the common difference of an arithmetic sequence.

The first 4 terms of this arithmetic sequence are the angles, measured in degrees, of quadrilateral Q

(c) Find the exact probability that the smallest angle of Q is more than 50°

(5)

(Total for question = 10 marks)

Q13.

The discrete random variable X has the following probability distribution

x	a	b	c
$P(X = x)$	$\log_{36} a$	$\log_{36} b$	$\log_{36} c$

where

- a, b and c are distinct integers (a < b < c)
- all the probabilities are greater than zero

(a) Find

(i) the value of a
(ii) the value of b
(iii) the value of c
Show your working clearly.

(5)

The independent random variables X_1 and X_2 each have the same distribution as X

(b) Find $P(X_1 = X_2)$

(2)

(Total for question = 7 marks)

Q14.

(a) State one disadvantage of using quota sampling compared with simple random sampling.

(1)

In a university 8% of students are members of the university dance club.

A random sample of 36 students is taken from the university.

The random variable X represents the number of these students who are members of the dance club.

(b) Using a suitable model for X, find

(i) $P(X = 4)$
(ii) $P(X \geq 7)$

(3)

Only 40% of the university dance club members can dance the tango.

(c) Find the probability that a student is a member of the university dance club and can dance the tango.

(1)

A random sample of 50 students is taken from the university.

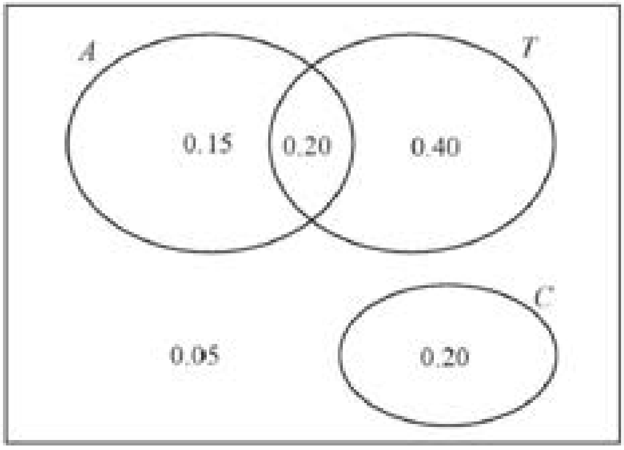
(d) Find the probability that fewer than 3 of these students are members of the university dance club and can dance the tango.

(2)

(Total for question = 7 marks)

Mark Scheme

Q1.

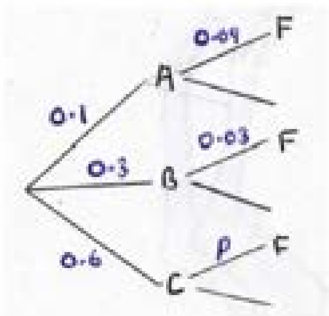
Question	Scheme	Marks	AOs
(a)	$p = [1 - 0.75 - 0.05 =]$ $\underline{0.20}$	B1	1.1b
		(1)	
(b)	$q = \underline{0.15}$	B1ft	1.1b
	$P(A) = 0.35$ $P(T) = 0.6$ $P(A \text{ and } T) = 0.20$ $P(A) \times P(T) = 0.21$	M1	2.1
	Since $0.20 \neq 0.21$ therefore A and T are **not** independent	A1	2.4
		(3)	
			
(c)	$P(\text{not } [A \text{ or } C]) = \underline{0.45}$	B1	1.1b
		(1)	
			(5 marks)

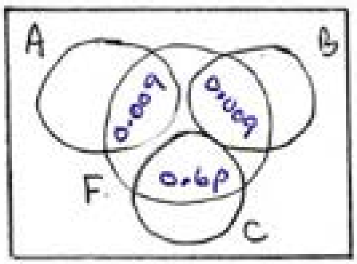
Part	Notes
(a)	B1cao for $p = 0.20$
(b)	B1ft for use of their p and $P(A \text{ or } T)$ to find q i.e. $0.75 - "p" - 0.40$ or $q = 0.15$
	M1 for the statement of all probabilities required for a suitable test and sight of any appropriate calculations required.
	A1 All probabilities correct, correct comparison and suitable comment.
(c)	B1cao for 0.45

Q2.

Qu	Scheme	Marks	AO
(a)	[Let $p = P(F \mid C)$]		
	Tree diagram or some other method to find an equation for p	M1	2.1
	$0.1 \times 0.09 + 0.3 \times 0.03 + 0.6 \times p = 0.06$	A1	1.1b
	$p = 0.07$ i.e. <u>7%</u>	A1	1.1b
		(3)	
(b)	e.g. $P(B \text{ and } F) = 0.3 \times 0.03 = 0.009$ but		
	$P(B) \times P(F) = 0.3 \times 0.06 = 0.018$	B1	2.4
	These are not equal so not independent		
		(1)	
		(4 marks)	

	Notes
(a)	M1 for selecting a suitable method to find the missing probability e.g. sight of tree diagram with 0.1, 0.3, 0.6 <u>and</u> 0.09, 0.03, p suitably placed e.g. sight of VD with 0.009 for $A \cap F$ and $B \cap F$ and $0.6p$ suitably placed <u>or</u> attempt an equation with at least one correct numerical and one "p" product (not necessarily correct) on LHS <u>or</u> for sight of $0.06 - (0.009 + 0.009)$ (o.e. e.g. $6 - 1.8 = 4.2\%$) 1st A1 for a correct equation for p (May be implied by a correct answer) <u>or</u> for the expression $\dfrac{0.06-(0.009+0.009)}{0.6}$ (o.e.) 2nd A1 for 7% (accept 0.07) **Correct Ans:** Provided there is no incorrect working seen award 3/3 e.g. may just see tree diagram with 0.07 for p (probably from trial and improv')
(b)	B1 for a suitable explanation…may talk about 2nd branches on tree diagram and point out that $0.03 \neq 0.06$ but need some supporting calculation/words Can condone incorrect use of set notation (it is not on AS spec) provided the rest of the calculations and words are correct.





Q3.

Qu	Scheme	Marks	AO
(a)	$P(X=4)=P(X=2)$ so $P(X=4)=0.35$	M1	2.1
	$P(X=1)=P(X=3)$ and $P(X=1)+P(X=3)=1-0.7$		
	So	A1	1.1b
	x: 1, 2, 3, 4; $P(X=x)$: 0.15, 0.35, 0.15, [0.35]		
		(2)	
(b)	Let A = number of spins that land on 4 $A\sim B(60, \text{"}0.35\text{"})$	B1ft	3.3
	$[P(A>30)=]\ 1-P(A\leqslant 30)$	M1	3.4
	$=1-0.99411\ldots\ =$ **awrt 0.00589**	A1	1.1b
		(3)	
(c)	$Y-X\leqslant 4 \ \Rightarrow \frac{12}{X}-X\leqslant 4$ or $12-X^2\leqslant 4X$ (since $X>0$) o.e.	M1	3.1a
	i.e. $0\leqslant X^2+4X-12 \Rightarrow 0\leqslant (X+6)(X-2)$ so $X\geqslant 2$	M1	1.1b
	$P(Y-X\leqslant 4)=P(X\geqslant 2)=0.35+0.15+0.35\ =\underline{0.85}$	A1	3.2a
		(3)	
		(8 marks)	

Notes

(a) M1 for using the given information to obtain $P(X=4)$

Award for statement $P(X=4)=P(X=2)$ <u>or</u> writing $P(X=4)=0.35$

A1 for getting fully correct distribution (any form that clearly identifies probs)

e.g. can be list $P(X=1)=0.15$, $P(X=3)=\ldots$etc

or as a probability function $P(X=x)=\begin{cases}0.15 & x=1,3\\ 0.35 & x=2,4\end{cases}$

[Condone missing $P(X=2)$ as this is given in QP]

(b) B1 for selecting a suitable model, sight of B(60, their 0.35) o.e. in words

f.t. their $P(X=4)$ from part (a).

Can be implied by $P(A\leqslant 30)=$ awrt 0.9941 or final answer = awrt 0.00589

M1 for using their model and interpreting "more than half"

Need to see $1-P(A\leqslant 30)$. Can be implied by awrt 0.00589

Can ignore incorrect LHS such as $P(A\geqslant 30)$

A1 for awrt 0.00589

(c) 1st M1 for translating the prob. problem into a <u>correct</u> mathematical inequality

Just an inequality in 1 variable. May be inside a probability statement.

ALT **Table of values:**

X	1	2	3	4
Y	12	6	4	3

<u>or</u> values of $Y-X=11, 4, 1, -1$

2nd M1 for solving the inequality leading to a range of values, allow 1 or 2 slips

May be a quadratic or cubic but must lead to a set of values of X or $Y-X$

ALT **Table or values:** They must state clearly which values are required

Both Ms can be implied by a correct answer (or correct ft of their distb'n)

A1 for interpreting the inequality and solving the problem i.e. 0.85 cao

Q4.

Question	Scheme	Marks	AOs
	$x = 0$	B1	2.2a
	$P(A) = 0.1 + z + y$ $\quad P(C) = 0.39 + z[+x]$ $\quad P(A \text{ and } C) = z$	M1	2.1
	$P(A \text{ and } C) = P(A) \times P(C) \rightarrow z = (0.1 + z + y) \times (0.39 + z[+x])$	M1	1.1b
	$\left[\sum p = 1\right]$ $0.06 + 0.3 + 0.39 + 0.1 + z + y[+x] = 1 \rightarrow \quad [z + y[+x] = 0.15]$	M1	1.1b
	Solving (simultaneously) leading to $\underline{z = 0.13}$ $\quad \underline{y = 0.02}$	A1	1.1b
			(5 marks)

	Notes
	B1: for $x = 0$, may be seen on Venn diagram
	M1: Identifying the probabilities required for independence and at least 2 correct These must be labelled If there are no labels, then this may be implied by $z = (0.1 + z + y)(0.39 + z\ [+x])$, allow one numerical slip Allow e.g. $P(A') = 0.39 + 0.30 + 0.06[+x]$ $\quad P(C) = 0.39 + z[+x]$ $\quad P(A' \text{ and } C) = 0.39$ [Not on spec. but you may see use of conditional probabilities]
	M1: Use of independence equation with their labelled probabilities in terms y, z [and x] All their probabilities must be substituted into a correct formula Sight of a correct equation e.g. $z = (0.1 + z + y)(0.39 + z\ [+x])$ scores M1M1
	M1: Using $\Sigma p = 1$ Implied by $[x +]\ y + z = 0.15$ or their $x + y + z = 0.15$ where x, y, and z are all probabilities or e.g. $P(A) = 0.25$
	A1: both $y = 0.02$ and $z = 0.13$

Q5.

Question	Scheme	Marks	AOs
(a)	(Discrete) uniform (distribution)	B1	1.2
		(1)	
(b)	B(28, 0.2)	B1	3.3
(i)	$P(X \geq 7) = 1 - P(X \leq 6)$ $[= 1 - 0.6784...]$	M1	3.4
	awrt **0.322**	A1	1.1b
(ii)	$P(4 \leq X < 8) = P(X \leq 7) - P(X \leq 3)$ $[= 0.818... - 0.160...]$	M1	3.1b
	awrt **0.658**	A1	1.1b
		(5)	
			(6 marks)

Notes

(a)	Continuous uniform is B0
(b)	**B1:** for identifying correct model, B(28, 0.2) allow B, bin or binomial may be implied by one correct answer or sight one correct probability i.e. awrt 0.678, awrt 0.818 or awrt 0.160 B(0.2, 28) is B0 unless it is used correctly
(i)	**M1:** Writing or using $1 - P(X \leq 6)$ or $1 - P(X < 7)$ **A1:** awrt 0.322 (correct answer only scores M1A1)
(ii)	**M1:** Writing or using $P(X \leq 7) - P(X \leq 3)$ or $P(X < 8) - P(X < 4)$ or $P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7)$ Condone P(4) as $P(X = 4)$, etc. **A1:** awrt 0.658 (correct answer only scores M1A1)

Q6.

Question	Scheme	Marks	AOs
	Overall method	M1	2.1
	$a+b=2c+0.5$ oe **or** $a+b=2(1-a-b)$	B1	2.2a
	$a+b+c=0.75$ oe	B1	1.1b
	$3c=0.25$ $\left[c=0.0833...\text{ or }\frac{1}{12}\right]$	M1	1.1b
	P(scoring 2,4 or 4,2 or 3,3) = $2\times"\frac{1}{12}"\times0.15+0.1^2$	M1	3.1b
	$=0.035$ oe	A1cso	1.1b
		(6)	
			(6 marks)

Notes		
	M1:	A fully correct method with all the required steps. For gaining 2 correct equations with at least one correct(allow if unsimplified). Attempting to solve to find a value of c followed by **correct method** to find the probability
	B1:	Forming a correct equation from the information given in the question
	B1:	A correct equation using the sum of the probabilities equals 1
	M1:	Correct method for solving 2 equations to find c Implied by $c=\frac{1}{12}$
	M1:	Recognising the ways to get a total of 6. Condone missing arrangments or repeats. Do not ignore extras written unless ignored in the calculation. May be implied by $m\times"\frac{1}{12}"\times0.15+n\times0.1^2$ where m and n are positive integers
	A1cso:	Cao 0.035, $\frac{7}{200}$ oe

Q7.

Question	Scheme	Marks	AOs
(a)	Let C = the number of successful calls. $C \sim B\left(9, \frac{1}{6}\right)$	M1	3.3
	$P(C \geq 3) = 1 - P(C \leq 2) = 0.1782...$ **awrt 0.178**	A1	1.1b
		(2)	
(b)	Let X = the number of occasions when at least 3 calls are successful. $P(X = 1) = 5 \times ("0.1782...") \times ("0.8217...")^4$	M1	1.1b
	$= 0.4061...$ **awrt 0.406**	A1	1.1b
		(2)	
(c)	$H_0 : p = \frac{1}{6}$ $H_1 : p > \frac{1}{6}$	B1	2.5
	Let R = the number of successful calls $R \sim B\left(35, \frac{1}{6}\right)$	M1	3.3
	$P(R \geq 11) = 1 - P(R \leq 10) = 0.02...$	A1	3.4
	There is sufficient evidence to support that **Rowan** has more successful sales calls than Afrika.	A1	2.2b
		(4)	
			(8 marks)

Notes		
(a)	M1:	For selecting the right model
	A1:	awrt 0.178
(b)	M1:	For $5 \times ("\text{their}(a)") \times ("1 - \text{their}(a)")^4$
	A1:	awrt 0.406
(c)	B1:	for correctly stating both hypotheses in terms of p or π Accept $p = 0.1\dot{6}$
	M1:	For selecting a suitable model. May be implied by a correct probability or CR
	A1:	Correct probability statement and answer of 0.02 or better (0.02318…) (CR $R \geq 11$ and either $P(R \leq 9) = 0.9450$ or $P(R \leq 10) = 0.9768$ or $1 - P(R \leq 10) = 0.0232$)
	A1:	Dependent on M1A1 but can ignore hypotheses. For conclusion in context supporting **Rowan's** belief / **Rowan** is a better sales person
		Do not accept Rowan can reject H_0

Q8.

Qu	Scheme	Marks	AO
(a)	[$p = 1 - (0.2 + 0.2 + 0.1 + 0.2)$] = $\underline{0.3}$	B1 (1)	1.1b
(b)	A and C are mutually exclusive. [NOT P(A) and P(C)]	B1 (1)	1.2
		(2 marks)	
	Notes		
(a)	B1 for		
(b)	B1 for A and C [NB $A \cap C$ or $A \cap C = \varnothing$ is B0] If more than one case given they must all be correct e.g. $A \cap B$ and C		

Q9.

Qu	Scheme	Marks	AO
	Must end up with 3 of each colour or 4 of each colour	M1	3.1b
	$\underline{n = 2}$ requires 1st red and 2nd green or red from A and green from B	M1	2.2a
	P(1st red and 2nd green) = $\frac{4}{9} \times \frac{1}{10} = \frac{4}{90}$ or $\frac{2}{45}$ $p = \underline{\frac{2}{45}}$	A1	1.1b
	$\underline{n = 5}$ requires 1st green and 2nd yellow or green from A and yellow from B	M1	2.2a
	P(1st green and 2nd yellow) = $\frac{5}{12} \times \frac{3}{10} = \frac{15}{120}$ or $\frac{1}{8}$ $p = \underline{\frac{1}{8}}$	A1	1.1b
		(5)	
		(5 marks)	
	Notes		
	1st M1 for an overall strategy realising there are 2 options. Award when evidence of both cases (3 of each colour or 4 of each colour) seen. 2nd M1 for $n = 2$ and attempt at 1st red and 2nd green May be implied by e.g. $\frac{4}{9} \times \frac{1}{9}$ 1st A1 for $p = \underline{\frac{2}{45}}$ or exact equivalent 3rd M1 for $n = 5$ and attempt at 1st green and 2nd yellow May be implied by e.g. $\frac{5}{12} \times \frac{3}{9}$ 2nd A1 for $p = \underline{\frac{1}{8}}$ or exact equivalent		
NB	If both correct values of p are found and then added (get $\frac{61}{360}$), deduct final A1 only (i.e. 4/5)		

Q10.

Qu	Scheme	Marks	AO
(a)	c: 0, 1, 2, 3, 4, 5, 6, 7, 8 $\mathrm{P}(C=c)$: $\frac{1}{9}$, $\frac{1}{9}$, $\frac{1}{9}$, $\frac{1}{9}$, $\frac{1}{9}$, $\frac{1}{9}$, $\frac{1}{9}$, $\frac{1}{9}$, $\frac{1}{9}$	B1 B1ft (2)	1.2 1.2
(b)	$\mathrm{P}(C<4)=\frac{4}{9}$ (accept 0.444 or better)	B1 (1)	3.4
(c)	Probability lower than expected suggests model is <u>not</u> good	B1ft (1)	3.5a
(d)	e.g. Cloud cover will vary from month to month and place to place So e.g. use a non-uniform distribution	B1 (1)	3.5c
		(5 marks)	

Notes

(a) 1st B1 for a correct set of values for c. Allow $\{\frac{1}{8}, \frac{2}{8}, \ldots \frac{8}{8}\}$

2nd B1ft for correct probs from their values for c, consistent with discrete uniform distrib'n Maybe as a prob. function. Allow $\mathrm{P}(X=x)=\frac{1}{9}$ for $0 \leqslant x \leqslant 8$ provided $x=\{0, 1, 2, \ldots, 8\}$ is clearly defined somewhere.

(b) B1 for using correct model to get $\frac{4}{9}$ (o.e.)

SC **Sample space** $\{1, \ldots, 8\}$ If scored B0B1 in (a) for this allow $\mathrm{P}(C<4)=\frac{3}{8}$ to score B1 in (b)

(c) B1ft for comment that states that the model proposed is or is not a good one based on their model in part (a) and their probability in (b)

$|(b)-0.315|>0.05$ Allow e.g. "it is not suitable"; "it is not accurate" etc

$|(b)-0.315| \leqslant 0.05$ Allow a comment that suggests it <u>is</u> suitable

No prob in (b) Allow a comparison that mentions 50% or 0.5 and rejects the model

No prob in (b) and no 50% or 0.5 or (b) > 1 scores B0

Ignore any comments about location or weather patterns.

(d) B1 for a sensible refinement considering variations in month or location

Just saying "not uniform" is B0

Context & "non-uniform" Allow mention of different locations, months <u>and</u> non-uniform <u>or</u> use more locations to form a new distribution with probabilities based on frequencies

Context & "binomial" Allow mention of different locations, months <u>and</u> binomial

Just refined model Model must be outlined and discrete and non-uniform e.g. higher probabilities for more cloud cover <u>or</u> lower probabilities for less cloud cover

Continuous model Any model that is based on a continuous distribution. e.g. normal is B0

Q11.

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$\frac{523+52+28}{184} = \frac{132}{184} = 0.717$	B1	This mark is given for a correct value for the probability for the cloud cover
(b)(i)	$P(X \geq 6) = 1 - P(X \leq 5)$	M1	This mark is given for using $1 - P(X \leq 5)$ with $B(8, 0.76)$
	$= 1 - 0.2967$ $= 0.703$	A1	This mark is given for finding as correct value for the probability
(b)(ii)	$184 \times P(X = 7)$ $= 184 \times 0.2811$	M1	This mark is given for using $184 \times P(X = 7)$ with $B(8, 0.76)$
	$= 51.7$	A1	This mark is given for finding as correct value for the probability
(c)	The answer to part (b)(i) of 0.703 is similar to 0.7127 in part (a) The answer to part (b)(ii) of 51.7 is very close to 52 found in the data set	B1	This mark is given for a correct evaluation of the outcomes from part (b) to determine the appropriateness of Magali's model
(d)	$\frac{5+9+9}{28} = \frac{23}{28} = 0.821$	B1	This mark is given for a correct value for the probability for the cloud cover
(e)	The answer to part (d) of 0.821 is greater than that in part (a) of 0.717 This shows that there is a higher chance of having high cloud cover if the previous day had high cloud cover	B1	This mark is given for a correct comparison for the answer to part (d) with the data set
	Thus independence does not hold so a binomial model might not be suitable	B1	This mark is given for a correct conclusion stated
			(Total 9 marks)

Q12.

	Scheme	Marks	AO
(a)	$\frac{k}{10}+\frac{k}{20}+\frac{k}{30}+\frac{k}{40}+\frac{k}{50}=1$ or $\frac{1}{600}(60k+30k+20k+15k+12k)=1$	M1	1.1b
	So $k=\frac{600}{137}$ (*)	A1cso	1.1b
		(2)	
(b)	(Cases are:) $D_1=30, D_2=50$ and $D_1=50, D_2=30$ and $D_1=40, D_2=40$	M1	2.1
	$\text{P}(D_1+D_2=80)=\frac{k}{50}\times\frac{k}{30}\times 2+\left(\frac{k}{40}\right)^2$	M1	3.4
	$=0.0375619\ldots$ awrt <u>**0.0376**</u>	A1	1.1b
		(3)	
(c)	Angles are: a, $a+d$, $a+2d$, $a+3d$	M1	3.1a
	$S_4=a+(a+d)+(a+2d)+(a+3d)=360$	M1	2.1
	$2a+3d=180$ (o.e.)	A1	2.2a
	Smallest angle is $a>50$ consider cases: $d=10$ so $a=75$ <u>or</u> $d=20$ so $a=60$ [$d=30$ gives $a=45$ no good]	M1	3.1b
	$\text{P}(D=10 \text{ or } 20)=\frac{3k}{20}=\underline{\frac{90}{137}}$	A1	1.1b
		(5)	
		(10 marks)	

	Notes
(a)	M1 for clear use of sum of probabilities = 1 (all terms seen) A1 cso (*) M1 scored and no incorrect working seen.
Verify	**(Assume** $k=\frac{600}{137}$**)** to score the final A1 they must have a <u>final</u> comment "$\therefore k=\frac{600}{137}$"
(b)	1st M1 for selecting at least 2 of the relevant cases (may be implied by their correct probs) e.g. allow 30, 50 and 50,30 i.e. D_1 and D_2 labels not required 2nd M1 for using the model to obtain a correct expression for two different probabilities. May use letter k or their value for k. Allow for $\frac{k}{50}\times\frac{k}{30}+\left(\frac{k}{40}\right)^2$ <u>or</u> $2\times\left(\frac{k}{50}\times\frac{k}{30}+\left(\frac{k}{40}\right)^2\right)$ A1 for awrt 0.0376 (exact fraction is $\frac{705}{18769}$)
(c)	1st M1 for recognising the 4 angles and finding expressions in terms of d and their a 2nd M1 for using property of quad with these 4 angles (equation can be un-simplified) Allow these two marks for use of a (possible) value of d e.g. $a+a+10+a+20+a+30=360$ (If at least 3 cases seen allow A1 for e.g. $4a=300$) <u>or</u> allow M1M1 for a set of 4 angles with sum 360 and possible value of d (3 cases for A1) e.g. (for $d=20$) 60, 80, 100, 120 1st A1 for $2a+3d=180$ condition (o.e.) [Must be in the form $pa+qd=N$] 3rd M1 for examining cases and getting $d=10$ and $d=20$ only 2nd A1 for $\frac{90}{137}$ or exact equivalent The correct answer and no obviously incorrect working will score 5/5 A final answer of awrt 0.657 (0.65693…) with no obviously incorrect working scores 4/5

Q13.

	Scheme	Marks	AO
(a)	[Sum of probs = 1 implies] $\log_{36} a + \log_{36} b + \log_{36} c = 1$	M1	3.1a
	$\Rightarrow \log_{36}(abc) = 1$ so $abc = 36$	A1	3.4
	All probabilities greater than 0 implies each of a, b and $c > 1$	B1	2.2a
	$36 = 2^2 \times 3^2$ (or 3 numbers that multiply to give 36 e.g. 2, 2, 9 etc)	dM1	2.1
	Since a, b and c are distinct must be <u>2, 3, 6</u> ($\underline{a = 2, b = 3, c = 6}$)	A1 (5)	3.2a
(b)	$(\log_{36} a)^2 + (\log_{36} b)^2 + (\log_{36} c)^2$	M1	3.4
	[= 0.0374137…+ 0.09398737…+0.25]		
	= 0.38140… awrt <u>0.381</u>	A1 (2)	1.1b
		(7 marks)	

	Notes
(a)	1st M1 for a start to the problem using sum of probabilities leading to eq'n in a, b and c
	1st A1 for reducing to the equation $abc = 36$ [Must follow from their equation.]
NB	Can go straight from $abc = 36$ to the answer for full marks for part (a).
	B1 for deducing that each value > 1 (may be implied by 3 integers all > 1 in the next line)
	2nd dM1 (dep on M1A1) for writing 36 as a product of prime factors <u>or</u> 3 values with product = 36 and none = 1
	2nd A1 for 2, 3 and 6 as a list or $a = 2$, $b = 3$ and $c = 6$
SC Ans only	M0M0 If no method marks scored but a correct answer given score: M0A0B1M0A1 (2/5) This gets the SC score of 2/5 [Question says show your working clearly]
(b)	M1 for a correct expression in terms of a, b and c or values; ft their integers a, b and c Condone invisible brackets if the answer implies they are used.
	A1 for awrt 0.381

Q14.

	Scheme	Marks	AO
(a)	**Disadvantage:** e.g. Not random; cannot use (reliably) for inferences	B1	1.1b
		(1)	
(b)	[Sight or correct use of] $X \sim B(36, 0.08)$	M1	3.3
(i)	$P(X = 4) = 0.167387\ldots$ awrt $\underline{0.167}$	A1	1.1b
(ii)	$\left[P(X \geqslant 7) = 1 - P(X \leqslant 6) =\right]$ $0.022233\ldots$ awrt $\underline{0.0222}$	A1	1.1b
		(3)	
(c)	P(In dance club and dance tango) = $0.4 \times 0.08 = \underline{0.032}$ or $\underline{\frac{4}{125}}$ or $\underline{3.2\%}$	B1	1.1b
		(1)	
(d)	[Let T = those who can dance the Tango. Sight or use of] $T \sim B(50, \text{"}0.032\text{"})$	M1	3.3
	$[P(T < 3) = P(T \leqslant 2) =]$ $0.7850815\ldots$ awrt $\underline{0.785}$	A1	1.1b
		(2)	
		(7 marks)	

Notes

(a) B1 for a suitable disadvantage:

Allow (B1)	Do NOT allow (B0)
Not random <u>or</u> less random (o.e.)	Not representative
Cannot use (reliably) for inferences	Less accurate
(More likely to be) biased	Any comment based on time or cost
	Any mention of skew
	Any mention of non-response

(b) M1 for sight of B(36, 0.08) Allow in words: <u>binomial</u> with <u>$n = 36$</u> and <u>$p = 0.08$</u>
may be implied by one correct answer to 2sf <u>or</u> sight of $P(X \leqslant 6) = 0.97776\ldots$ i.e. awrt 0.98
Allow for $36C4 \times 0.08^4 \times 0.92^{32}$ as this is "correct use"

(i) 1st A1 for awrt 0.167 NB An answer of just awrt 0.167 scores M1($\Rightarrow$)1st A1

(ii) 2nd A1 for awrt 0.0222

(c) B1 for 0.032 o.e. (Can allow for sight of 0.4×0.08)

(d) M1 for sight of B(50, "0.032") ft their answer to (c) provided it is a probability $\neq 0.08$
may be implied by correct answer
<u>or</u> sight of $[P(T \leqslant 3)] = 0.924348\ldots$ i.e. awrt 0.924 or $P(T \leqslant 2)$ as part of $1 - P(T \leqslant 2)$ calc.
A1 for awrt 0.785

MR Allow MR of 50 (e.g. 30) provided clearly attempting $P(T \leqslant 2)$ and score M1A0

Year 2: A Level Mathematics	
<u>Statistics: Statistical Distributions</u>	
Self-Assessment: Please identify areas in which you believe are your strong points and those you feel you need to improve on Provide evidence to support your assessment with reference to the content in this booklet.	
Strengths	**Areas for Improvement**

Questions

Q1.

A nursery has a sack containing a large number of coloured beads of which 14% are coloured red.

Aliya takes a random sample of 18 beads from the sack to make a bracelet.

(a) State a suitable binomial distribution to model the number of red beads in Aliya's bracelet.

(1)

(b) Use this binomial distribution to find the probability that

(i) Aliya has just 1 red bead in her bracelet,
(ii) there are at least 4 red beads in Aliya's bracelet.

(3)

(c) Comment on the suitability of a binomial distribution to model this situation.

(1)

After several children have used beads from the sack, the nursery teacher decides to test whether or not the proportion of red beads in the sack has changed.
She takes a random sample of 75 beads and finds 4 red beads.

(d) Stating your hypotheses clearly, use a 5% significance level to carry out a suitable test for the teacher.

(4)

(e) Find the *p*-value in this case.

(1)

(Total for question = 10 marks)

Q2.

In an experiment a group of children each repeatedly throw a dart at a target.
For each child, the random variable H represents the number of times the dart hits the target in the first 10 throws.

Peta models H as B(10, 0.1)

(a) State two assumptions Peta needs to make to use her model.

(2)

(b) Using Peta's model, find $P(H \geq 4)$

(1)

For each child the random variable F represents the number of the throw on which the dart first hits the target.

Using Peta's assumptions about this experiment,

(c) find $P(F = 5)$

(2)

Thomas assumes that in this experiment no child will need more than 10 throws for the dart to hit the target for the first time. He models $P(F = n)$ as

$$P(F = n) = 0.01 + (n - 1) \times \alpha$$

where α is a constant.

(d) Find the value of α

(4)

(e) Using Thomas' model, find $P(F = 5)$

(1)

(f) Explain how Peta's and Thomas' models differ in describing the probability that a dart hits the target in this experiment.

(1)

(Total for question = 11 marks)

Q3.

Magali is studying the mean total cloud cover, in oktas, for Leuchars in 1987 using data from the large data set. The daily mean total cloud cover for all 184 days from the large data set is summarised in the table below.

Daily mean total cloud cover (oktas)	0	1	2	3	4	5	6	7	8
Frequency (number of days)	0	1	4	7	10	30	52	52	28

One of the 184 days is selected at random.

(a) Find the probability that it has a daily mean total cloud cover of 6 or greater.

(1)

Magali is investigating whether the daily mean total cloud cover can be modelled using a binomial distribution.

She uses the random variable X to denote the daily mean total cloud cover and believes that $X \sim B(8, 0.76)$

Using Magali's model,

(b) (i) find $P(X \geq 6)$

(2)

(ii) find, to 1 decimal place, the expected number of days in a sample of 184 days with a daily mean total cloud cover of 7

(2)

(c) Explain whether or not your answers to part (b) support the use of Magali's model.

(1)

There were 28 days that had a daily mean total cloud cover of 8
For these 28 days the daily mean total cloud cover for the **following** day is shown in the table below.

Daily mean total cloud cover (oktas)	0	1	2	3	4	5	6	7	8
Frequency (number of days)	0	0	1	1	2	1	5	9	9

(d) Find the proportion of these days when the daily mean total cloud cover was 6 or greater.

(1)

(e) Comment on Magali's model in light of your answer to part (d).

(2)

(Total for question = 9 marks)

Q4.

(a) State one disadvantage of using quota sampling compared with simple random sampling.

(1)

In a university 8% of students are members of the university dance club.

A random sample of 36 students is taken from the university.

The random variable X represents the number of these students who are members of the dance club.

(b) Using a suitable model for X, find

(i) $P(X = 4)$
(ii) $P(X \geq 7)$

(3)

Only 40% of the university dance club members can dance the tango.

(c) Find the probability that a student is a member of the university dance club and can dance the tango.

(1)

A random sample of 50 students is taken from the university.

(d) Find the probability that fewer than 3 of these students are members of the university dance club and can dance the tango.

(2)

(Total for question = 7 marks)

Mark Scheme

Q1.

Qu	Scheme	Marks	AO
(a)	[R = no. of red beads in Aliya's bracelet] $R \sim B(18, 0.14)$	B1 (1)	3.3
(b)(i)	$P(R = 1) = 0.19403...$ awrt **0.194**	B1	1.1b
(ii)	$P(R \geqslant 4) = 1 - P(R \leqslant 3) = 1 - [0.76184...]$	M1	3.4
	$= 0.2381588...$ awrt **0.238**	A1 (3)	1.1b
(c)	Requires $p = 0.14$ to be constant so need a large number of beads in the sack to ensure that removing 18 beads does not appreciably affect this probability, then it could be suitable.	B1 (1)	3.5b
(d)	$H_0 : p = 0.14$ $H_1 : p \neq 0.14$	B1	2.5
	[X = number of red beads in the sample] $X \sim B(75, 0.14)$	M1	3.3
	$P(X \leqslant 4) = 0.01506...$ or if B(75, 0.14) seen awrt 0.02	A1	3.4
	{0.02 < 0.025 so significant or reject H_0} There is evidence that the proportion of red beads has changed	A1 (4)	2.2b
(e)	p-value is $2 \times$"0.01506..."$= 0.030123...$ = awrt 0.03	B1ft (1)	1.1b
		(10 marks)	

	Notes
(a)	B1 for B(18, 0.14) accept in words e.g. binomial with $n = 18$ and $p = 0.14$
(b)(i)	B1 for awrt 0.194
(ii)	M1 for interpreting "at least 4" Need $1 - P(R \leqslant 3)$ and $1 - p$ $[0 < p < 1]$ $P(R = 3) = 0.233..$ OK
	A1 for awrt 0.238
(c)	B1 for mention of large number of beads and need for $p = 0.14$ to be constant for it to be suitable. Do NOT accept e.g. "events are independent"
(d)	B1 for both hypotheses correct with use of p or π
	M1 for selecting a suitable model: sight or correct use of B(75, 0.14) May be implied by sight of 0.015 or better or $[P(X > 4) =]$ 0.9849..., i.e. 0.985 or better
	1st A1 for use of the correct model awrt 0.015 (accept awrt 0.02 following a correct expression) Allow 1st A1 for awrt 0.985 only if correct comparison with 0.975 is seen. Sight of B(75, 0.14) and $P(X \leqslant 4)$ = awrt 0.02 scores M1A1
	No sight of B(75, 0.14) but sight of awrt 0.015 scores M1($\Rightarrow$)A1[Condone $P(X = 4) = ..$]
	2nd A1 (dep on M1A1) for a correct conclusion in context mentioning "proportion", "red" and "changed" If there is a statement about H_0 or significance it must be compatible.
NB	May see CR i.e. $X \leqslant 4$ (mark when prob seen) and $X \geqslant 18$ (prob = 0.01406..) Ignore upper limit
	NB for information $P(X = 4) = 0.0104...$ and can only score M1A0A0 if B(75, 0.14) seen
(e)	B1ft for awrt 0.03 Allow ft of their probability in (d) provided at least 3sf used NB an answer of 0.02 in (d) leading to 0.04 in (e) is B0
SC	Use of CR will give significance level of 0.01506...+ 0.01406... = 0.029... score B1 no ft

Q2.

Qu	Scheme	Marks	AO
(a)	The probability of a dart hitting the target is constant (from child to child and for each throw by each child) (o.e.)	B1	1.2
	The throws of each of the darts are independent (o.e.)	B1	1.2
		(2)	
(b)	[P($H \geqslant 4$) = 1 − P($H \leqslant 3$) = 1 − 0.9872 = 0.012795.. =] awrt 0.0128	B1	1.1b
		(1)	
(c)	P(F = 5) = $0.9^4 \times 0.1$, = 0.06561	M1,	3.4
	= awrt 0.0656	A1	1.1b
		(2)	
(d)	<table><tr><td>n</td><td>1</td><td>2</td><td>…</td><td>10</td></tr><tr><td>P($F = n$)</td><td>0.01</td><td>0.01 + α</td><td>…</td><td>0.01+9α</td></tr></table>	M1	3.1b
	Sum of probs = 1 $\Rightarrow \frac{10}{2}[2\times0.01+9\alpha]=1$	M1A1	3.1a 1.1b
	[i.e. $5(0.02+9\alpha)=1$ or $0.1+45\alpha=1$] so α = 0.02	A1	1.1b
		(4)	
(e)	P(F = 5| Thomas' model) = 0.09	B1ft	3.4
		(1)	
(f)	Peta's model assumes the probability of hitting target is constant (o.e.) and Thomas' model assumes this probability increases with each attempt(o.e.)	B1	3.5a
		(1)	
		(11 marks)	

	Notes
(a)	1st B1 for stating that the probability (or possibility or chance) is constant (or fixed or same) 2nd B1 for stating that throws are independent ["trials" are independent is B0]
(b)	B1 for awrt 0.0128 (found on calculator)
(c)	M1 for a probability expression of the form $(1-p)^4 \times p$ where $0 < p < 1$ A1 for awrt 0.0656
SC	Allow M1A0 for answer only of 0.066
(d)	1st M1 for setting up the distribution of F with at least 3 correct values of n and P($F = n$) in terms of α. (Can be implied by 2nd M1 or 1st A1) 2nd M1 for use of sum of probs = 1 and clear summation or use of arithmetic series formula (allow 1 error or missing term). (Can be implied by 1st A1) 1st A1 for a correct equation for α 2nd A1 for α = 0.02 (must be exact and come from correct working)
(e)	B1ft for value resulting from $0.01 + 4\times$"their α" (provided α and the answer are probs) **Beware** If their answer is the same as their (c) (or a rounded version of their (c)) score B0
(f)	B1 for a suitable comment about the probability of hitting the target
ALT	Allow idea that Peta's model suggests the dart may never hit the target but Thomas' says that it will hit at least once (in the first 10 throws).

Q3.

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$\frac{523+52+28}{184} = \frac{132}{184} = 0.717$	B1	This mark is given for a correct value for the probability for the cloud cover
(b)(i)	$P(X \geq 6) = 1 - P(X \leq 5)$	M1	This mark is given for using $1 - P(X \leq 5)$ with $B(8, 0.76)$
	$= 1 - 0.2967$ $= 0.703$	A1	This mark is given for finding as correct value for the probability
(b)(ii)	$184 \times P(X = 7)$ $= 184 \times 0.2811$	M1	This mark is given for using $184 \times P(X = 7)$ with $B(8, 0.76)$
	$= 51.7$	A1	This mark is given for finding as correct value for the probability
(c)	The answer to part (b)(i) of 0.703 is similar to 0.7127 in part (a) The answer to part (b)(ii) of 51.7 is very close to 52 found in the data set	B1	This mark is given for a correct evaluation of the outcomes from part (b) to determine the appropriateness of Magali's model
(d)	$\frac{5+9+9}{28} = \frac{23}{28} = 0.821$	B1	This mark is given for a correct value for the probability for the cloud cover
(e)	The answer to part (d) of 0.821 is greater than that in part (a) of 0.717 This shows that there is a higher chance of having high cloud cover if the previous day had high cloud cover	B1	This mark is given for a correct comparison for the answer to part (d) with the data set
	Thus independence does not hold so a binomial model might not be suitable	B1	This mark is given for a correct conclusion stated
			(Total 9 marks)

Q4.

	Scheme	Marks	AO
(a)	**Disadvantage:** e.g. Not random; cannot use (reliably) for inferences	B1	1.1b
		(1)	
(b)	[Sight or correct use of] $X \sim B(36, 0.08)$	M1	3.3
(i)	$P(X = 4) = 0.167387...$ awrt <u>0.167</u>	A1	1.1b
(ii)	$[P(X \geqslant 7) = 1 - P(X \leqslant 6) =]$ $0.022233...$ awrt <u>0.0222</u>	A1	1.1b
		(3)	
(c)	P(In dance club and dance tango) = 0.4×0.08 = <u>0.032</u> or $\frac{4}{125}$ or <u>3.2%</u>	B1	1.1b
		(1)	
(d)	[Let T = those who can dance the Tango. Sight or use of] $T \sim B(50, \text{"}0.032\text{"})$	M1	3.3
	$[P(T < 3) = P(T \leqslant 2) =]$ $0.7850815...$ awrt <u>0.785</u>	A1	1.1b
		(2)	
		(7 marks)	

Notes

(a) B1 for a suitable disadvantage:

Allow (B1)	Do NOT allow (B0)
Not random <u>or</u> less random (o.e.)	Not representative
Cannot use (reliably) for inferences	Less accurate
(More likely to be) biased	Any comment based on time or cost
	Any mention of skew
	Any mention of non-response

(b) M1 for sight of B(36, 0.08) Allow in words: <u>binomial</u> with <u>$n = 36$</u> and <u>$p = 0.08$</u>
may be implied by one correct answer to 2sf <u>or</u> sight of $P(X \leqslant 6) = 0.97776...$ i.e. awrt 0.98
Allow for $36C4 \times 0.08^4 \times 0.92^{32}$ as this is "correct use"

(i) 1st A1 for awrt 0.167 NB An answer of just awrt 0.167 scores M1($\Rightarrow$)1st A1

(ii) 2nd A1 for awrt 0.0222

(c) B1 for 0.032 o.e. (Can allow for sight of 0.4×0.08)

(d) M1 for sight of B(50, "0.032") ft their answer to (c) provided it is a probability $\neq 0.08$
may be implied by correct answer
<u>or</u> sight of $[P(T \leqslant 3)] = 0.924348...$ i.e. awrt 0.924 or $P(T \leqslant 2)$ as part of $1 - P(T \leqslant 2)$ calc.
A1 for awrt 0.785

MR Allow MR of 50 (e.g. 30) provided clearly attempting $P(T \leqslant 2)$ and score M1A0

Year 2: A Level Mathematics	
<u>Statistics: Hypothesis Testing</u>	
Self-Assessment: Please identify areas in which you believe are your strong points and those you feel you need to improve on Provide evidence to support your assessment with reference to the content in this booklet.	
Strengths	**Areas for Improvement**

Questions

Q1.

(a) The discrete random variable $X \sim B(40, 0.27)$

Find $P(X \geq 16)$

(2)

Past records suggest that 30% of customers who buy baked beans from a large supermarket buy them in single tins. A new manager suspects that there has been a change in the proportion of customers who buy baked beans in single tins. A random sample of 20 customers who had bought baked beans was taken.

(b) Write down the hypotheses that should be used to test the manager's suspicion.

(1)

(c) Using a 10% level of significance, find the critical region for a two-tailed test to answer the manager's suspicion. You should state the probability of rejection in each tail, which should be less than 0.05

(3)

(d) Find the actual significance level of a test based on your critical region from part (c).

(1)

One afternoon the manager observes that 12 of the 20 customers who bought baked beans, bought their beans in single tins.

(e) Comment on the manager's suspicion in the light of this observation.

(1)

Later it was discovered that the local scout group visited the supermarket that afternoon to buy food for their camping trip.

(f) Comment on the validity of the model used to obtain the answer to part (e), giving a reason for your answer.

(1)

(Total for question = 9 marks)

Q2.

Naasir is playing a game with two friends. The game is designed to be a game of chance so that the probability of Naasir winning each game is $\frac{1}{3}$

Naasir and his friends play the game 15 times.

(a) Find the probability that Naasir wins

(i) exactly 2 games,
(ii) more than 5 games.

(3)

Naasir claims he has a method to help him win more than $\frac{1}{3}$ of the games. To test this claim, the three of them played the game again 32 times and Naasir won 16 of these games.

(b) Stating your hypotheses clearly, test Naasir's claim at the 5% level of significance.

(4)

(Total for question = 7 marks)

Q3.

Past records show that 15% of customers at a shop buy chocolate. The shopkeeper believes that moving the chocolate closer to the till will increase the proportion of customers buying chocolate.

After moving the chocolate closer to the till, a random sample of 30 customers is taken and 8 of them are found to have bought chocolate.

Julie carries out a hypothesis test, at the 5% level of significance, to test the shopkeeper's belief.

Julie's hypothesis test is shown below.

$H_0 : p = 0.15$
$H_1 : p \geq 0.15$
Let X = the number of customers who buy chocolate.
$X \sim B(30, 0.15)$
$P(X = 8) = 0.0420$
$0.0420 < 0.05$ so reject H_0
There is sufficient evidence to suggest that the proportion of customers buying chocolate has increased.

(a) Identify the first two errors that Julie has made in her hypothesis test.

(2)

(b) Explain whether or not these errors will affect the conclusion of her hypothesis test.

Give a reason for your answer.

(1)

(c) Find, using a 5% level of significance, the critical region for a one-tailed test of the shopkeeper's belief. The probability in the tail should be less than 0.05

(2)

(d) Find the actual level of significance of this test.

(1)

(Total for question = 6 marks)

Q4.

Afrika works in a call centre.

She assumes that calls are independent and knows, from past experience, that on each sales call

that she makes there is a probability of $\frac{1}{6}$ that it is successful.

Afrika makes 9 sales calls.

(a) Calculate the probability that at least 3 of these sales calls will be successful.

(2)

The probability of Afrika making a successful sales call is the same each day.

Afrika makes 9 sales calls on each of 5 different days.

(b) Calculate the probability that at least 3 of the sales calls will be successful on exactly 1 of these days.

(2)

Rowan works in the same call centre as Afrika and believes he is a more successful salesperson.

To check Rowan's belief, Afrika monitors the next 35 sales calls Rowan makes and finds that 11 of the sales calls are successful.

(c) Stating your hypotheses clearly test, at the 5% level of significance, whether or not there is evidence to support Rowan's belief.

(4)

(Total for question = 8 marks)

Q5.

A nursery has a sack containing a large number of coloured beads of which 14% are coloured red.

Aliya takes a random sample of 18 beads from the sack to make a bracelet.

(a) State a suitable binomial distribution to model the number of red beads in Aliya's bracelet.

(1)

(b) Use this binomial distribution to find the probability that

(i) Aliya has just 1 red bead in her bracelet,
(ii) there are at least 4 red beads in Aliya's bracelet.

(3)

(c) Comment on the suitability of a binomial distribution to model this situation.

(1)

After several children have used beads from the sack, the nursery teacher decides to test whether or not the proportion of red beads in the sack has changed.
She takes a random sample of 75 beads and finds 4 red beads.

(d) Stating your hypotheses clearly, use a 5% significance level to carry out a suitable test for the teacher.

(4)

(e) Find the *p*-value in this case.

(1)

(Total for question = 10 marks)

Mark Scheme

Q1.

Question	Scheme	Marks	AOs
(a)	$P(X \geqslant 16) = 1 - P(X \leqslant 15)$	M1	1.1b
	$= 1 - 0.949077...$ $\quad = $ awrt $\underline{0.0509}$	A1	1.1b
		(2)	
(b)	$H_0 : p = 0.3 \quad H_1 : p \neq 0.3$ (Both correct in terms of p or π)	B1	2.5
		(1)	
(c)	[$Y \sim B(20, 0.3)$] sight of $P(Y \leqslant 2) = 0.0355$ or $P(Y \leqslant 9) = 0.9520$	M1	2.1
	Critical region is $\{Y \leqslant 2\}$ or (o.e.)	A1	1.1b
	$\{Y \geqslant 10\}$ (o.e.)	A1	1.1b
		(3)	
(d)	[0.0355 + (1 – 0.9520)] = 0.0835 <u>or</u> <u>8.35%</u>	B1ft	1.1b
		(1)	
(e)	(Assuming that the 20 customers represent a random sample then) 12 is in the CR so the manager's suspicion is supported	B1ft	3.2a
		(1)	
(f)	e.g. (e) requires the 20 customers to be a random sample or independent and the members of the scout group may invalidate this so binomial distribution would not be valid (and conclusion in (e) is probably not valid)	B1	3.5a
		(1)	
			(9 marks)

Part	Notes
(a)	M1 for dealing with $P(X \geqslant 16)$ – they need to use cumulative prob. function on calc.
	A1 awrt 0.0509 (from calculator)
(b)	B1 for both hypotheses in terms of p or π and H_1 must be 2-tail
(c)	M1 for correct use of tables to find probability associated with critical value.
	1st A1 for the correct lower limit of the CR. Do not award for $P(Y \leqslant 2)$
	2nd A1 for the correct upper limit.
(d)	B1ft ft on their 0.0355 and (1 – their 0.9520) provided each probability is less than 0.05
(e)	B1ft for a comment that relates 12 to their CR and makes a consistent comment relating this to the manager's suspicion
(f)	B1 for a comment that: gives a suitable reason based on lack of independence <u>or</u> the sample not being random <u>so</u> the binomial model is not valid

Q2.

Qu	Scheme	Marks	AO
(a)	Let N = the number of games Naasir wins $N \sim B(15, \frac{1}{3})$	M1	3.3
(i)	$P(N = 2) = 0.059946...$ **awrt 0.0599**	A1	1.1b
(ii)	$P(N > 5) = 1 - P(N \leqslant 5) = 0.38162...$ **awrt 0.382**	A1	1.1b
		(3)	
(b)	$H_0 : p = \frac{1}{3}$ $H_1 : p > \frac{1}{3}$	B1	2.5
	Let X = the number of games Naasir wins $X \sim B(32, \frac{1}{3})$	M1	3.3
	$P(X \geqslant 16) = 1 - P(X \leqslant 15) = 0.03765$ (< 0.05)	A1	3.4
	[Significant result so reject H_0 (the null model) and conclude:] There is evidence to support Naasir's claim (o.e.)	A1	3.5a
		(4)	
		(7 marks)	

	Notes
(a)	M1 for selecting a binomial model with correct n and p Award for sight of $B(15, \frac{1}{3})$ (o.e. e.g. in words) or implied by 1 correct answer 1st A1 for awrt 0.0599 (from a calculator). Allow 0.05995 2nd A1 for awrt 0.382 (from a calculator)
(b)	B1 for correctly stating both hypotheses in terms of p or π Accept $p = 0.\dot{3}$ or any exact equivalent. $H_1 : p \geqslant \frac{1}{3}$ is B0 M1 for selecting a suitable model to use for the test. Award for sight of $B(32, \frac{1}{3})$ (o.e. e.g. in words) or implied by 0.03765 Can also allow M1 for $P(X \leqslant 15) = 0.962$ or better or $P(X \leqslant 14) = 0.922$ or better 1st A1 for use of the model to calculate an appropriate probability using calc. Sight of $P(X \geqslant 16)$ **and** answer awrt 0.0377
ALT	**CR** May use CR so award 1st A1 for CR of $X \geqslant 16$ must have seen some probabilities though: 1 of $P(X \leqslant 15) = 0.9623$ or $P(X \leqslant 14) = 0.9224$ or 0.9223 2nd A1 for conclusion in context that there is support for Naasir's claim Must mention "Naasir" or "his" and "claim" or "method" (o.e.) or e.g. probability of winning a game is $> \frac{1}{3}$ or has increased Dependent on M1 and 1st A1 but can ignore hypotheses but see below If you see $P(X \geqslant 16) = 0.0376$ followed by a correct contextualised conclusion then please award A0A1
SC	**Use of 0.3 for $\frac{1}{3}$** If used 0.3 instead of $\frac{1}{3}$ in (a) and score M0A0A0 can condone use of 0.3 in (b) 1st A1 ft needs $P(X \geqslant 16) = 0.0138$ or CR of $X \geqslant 15$ and sight of 1 of $P(X \geqslant 15) = 0.0327$ or $P(X \geqslant 14) = 0.0694$ 2nd A1 as before with 0.3 instead $\frac{1}{3}$ (if appropriate)

Q3.

Question	Scheme	Marks	AOs
(a)	The alternative hypothesis should be $H_1: p > 0.15$	B1	2.5
	The calculation of the test statistic should be $P(X \geq 8)$ [= 0.0698]	B1	2.3
		(2)	
(b)	These will affect the conclusion (as the null hypothesis should not be rejected) since $P(X \geq 8)$ [= 0.0698] is greater than 0.05	B1	2.4
		(1)	
(c)	$P(X \leq 8) = 0.9722... > 0.95$ or $P(X \geq 9) = 0.0277... < 0.05$	M1	2.1
	CR: $\{X \geq 9\}$	A1	1.1b
		(2)	
(d)	awrt **0.0278**	B1ft	1.1b
		(1)	
			(6 marks)

Notes	
(a)	**B1:** Identifying that $\geq$ should be $>$ in the alternative hypothesis **B1:** Identifying that $P(X = 8)$ should be $P(X \geq 8)$ Stating $P(X = 8)$ is incorrect on its own is insufficient Check for errors identified and corrected next to the question
(b)	**B1:** Will affect conclusion **and** correct supporting reason
(c)	**M1:** For use of tables to find probability associated with critical value [$P(X \leq 8)$ or $P(X \geq 9)$ with B(30, 0.15) (may be implied by either correct probability awrt 0.97 or awrt 0.03) or by the correct CR] **A1:** $[30 \geq] X \geq 9$ o.e. e.g. $X > 8$ Allow '9 or more' or 'CR $\geq$ 9'
(d)	**B1ft:** awrt 0.0278 (allow awrt 2.78%) or correct ft their one-tailed upper CR from B(30, 0.15) to 3s.f.

Q4.

Question	Scheme	Marks	AOs
(a)	Let C = the number of successful calls. $C \sim B\left(9, \frac{1}{6}\right)$	M1	3.3
	$P(C \geq 3) = 1 - P(C \leq 2) = 0.1782...$ **awrt 0.178**	A1	1.1b
		(2)	
(b)	Let X = the number of occasions when at least 3 calls are successful. $P(X = 1) = 5 \times (\text{"}0.1782...\text{"}) \times (\text{"}0.8217...\text{"})^4$	M1	1.1b
	$= 0.4061...$ **awrt 0.406**	A1	1.1b
		(2)	
(c)	$H_0 : p = \frac{1}{6}$ $H_1 : p > \frac{1}{6}$	B1	2.5
	Let R = the number of successful calls $R \sim B\left(35, \frac{1}{6}\right)$	M1	3.3
	$P(R \geq 11) = 1 - P(R \leq 10) = 0.02...$	A1	3.4
	There is sufficient evidence to support that **Rowan** has more successful sales calls than Afrika.	A1	2.2b
		(4)	
			(8 marks)

Notes		
(a)	**M1:**	For selecting the right model
	A1:	awrt 0.178
(b)	**M1:**	For $5 \times (\text{"their}(a)\text{"}) \times (\text{"}1 - \text{their}(a)\text{"})^4$
	A1:	awrt 0.406
(c)	**B1:**	for correctly stating both hypotheses in terms of p or π Accept $p = 0.1\dot{6}$
	M1:	For selecting a suitable model. May be implied by a correct probability or CR
	A1:	Correct probability statement and answer of 0.02 or better (0.02318…) (CR $R \geq 11$ and either $P(R \leq 9) = 0.9450$ or $P(R \leq 10) = 0.9768$ or $1 - P(R \leq 10) = 0.0232$)
	A1:	Dependent on M1A1 but can ignore hypotheses. For conclusion in context supporting **Rowan's belief / Rowan is a better sales person**
		Do not accept Rowan can reject H_0

Q5.

Qu	Scheme	Marks	AO
(a)	[R = no. of red beads in Aliya's bracelet] $R \sim \mathrm{B}(18, 0.14)$	B1 (1)	3.3
(b)(i)	$\mathrm{P}(R = 1) = 0.19403\ldots$ awrt **0.194**	B1	1.1b
(ii)	$\mathrm{P}(R \geqslant 4) = 1 - \mathrm{P}(R \leqslant 3) = 1 - [0.76184\ldots]$	M1	3.4
	$= 0.2381588\ldots$ awrt **0.238**	A1 (3)	1.1b
(c)	Requires $p = 0.14$ to be constant so need a large number of beads in the sack to ensure that removing 18 beads does not appreciably affect this probability, then it could be suitable.	B1 (1)	3.5b
(d)	$\mathrm{H}_0 : p = 0.14$ $\mathrm{H}_1 : p \neq 0.14$	B1	2.5
	[X = number of red beads in the sample] $X \sim \mathrm{B}(75, 0.14)$	M1	3.3
	$\mathrm{P}(X \leqslant 4) = 0.01506\ldots$ or if $\mathrm{B}(75, 0.14)$ seen awrt 0.02	A1	3.4
	{0.02 < 0.025 so significant or reject H_0} There is evidence that the proportion of red beads has changed	A1 (4)	2.2b
(e)	p-value is $2\times$"0.01506..." = 0.030123... = awrt 0.03	B1ft (1)	1.1b
		(10 marks)	

	Notes
(a)	B1 for B(18, 0.14) accept in words e.g. binomial with $n = 18$ and $p = 0.14$
(b)(i)	B1 for awrt 0.194
(ii)	M1 for interpreting "at least 4" Need $1 - \mathrm{P}(R \leqslant 3)$ and $1 - p$ $[0 < p < 1]$ $\mathrm{P}(R = 3) = 0.233..$ OK A1 for awrt 0.238
(c)	B1 for mention of large number of beads and need for $p = 0.14$ to be constant for it to be suitable. Do NOT accept e.g. "events are independent"
(d)	B1 for both hypotheses correct with use of p or π M1 for selecting a suitable model: sight or correct use of B(75, 0.14) May be implied by sight of 0.015 or better or $[\mathrm{P}(X > 4) =]$ 0.9849… i.e. 0.985 or better 1st A1 for use of the correct model awrt 0.015 (accept awrt 0.02 following a correct expression) Allow 1st A1 for awrt 0.985 only if correct comparison with 0.975 is seen. Sight of B(75, 0.14) and $\mathrm{P}(X \leqslant 4)$ = awrt 0.02 scores M1A1 No sight of B(75, 0.14) but sight of awrt 0.015 scores M1($\Rightarrow$)A1[Condone $\mathrm{P}(X = 4) = ..$] 2nd A1 (dep on M1A1) for a correct conclusion in context mentioning "proportion", "red" and "changed" If there is a statement about H_0 or significance it must be compatible.
NB	May see CR i.e. $X \leqslant 4$ (mark when prob seen) and $X \geqslant 18$ (prob = 0.01406..) Ignore upper limit NB for information $\mathrm{P}(X = 4) = 0.0104\ldots$ and can only score M1A0A0 if B(75, 0.14) seen
(e)	B1ft for awrt 0.03 Allow ft of their probability in (d) provided at least 3sf used NB an answer of 0.02 in (d) leading to 0.04 in (e) is B0
SC	Use of CR will give significance level of 0.01506…+ 0.01406… = 0.029… score B1 no ft